Quantum Hall Fractions in Rotating Bose-Einstein Condensates

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We study the quantum Hall phases that appear in the dilute limit of rotating Bose-Einstein condensates. By exact diagonalization in a spherical geometry we obtain the ground state and low-lying excited states of a small number of bosons as a function of the filling fraction ν, the ratio of the number of bosons to the number of vortices. We show the occurrence of the Jain principal sequence of incompressible liquids for $\nu = 2/3, 3/4, 4/3, 5/4$ in addition to the Laughlin state $\nu = 1/2$ as well as the Pfaffian state for $\nu = 1$. We give gap estimates by finite-size scaling of both charged and neutral excitations.

Bose-Einstein condensates in dilute atomic gases offer a unique opportunity to investigate the physics of vortex matter when they undergo rotation [1,2]. Indeed, recent experiments [3,4] have observed the appearance of large vortex arrays at sufficient high angular velocity ω. In addition to this phase akin to the Abrikosov lattice of type-II superconductors, there is the possibility that at larger ω the lattice melts [5,6] and is replaced by a quantum Hall liquid. Consider a trap with strong confinement in the z direction such that the system is effectively two-dimensional (2D). Then if the rotation frequency is tuned to the characteristic frequency of the harmonic confining potential in the xy plane, the bosons feel only the Coriolis force and the system is equivalent to 2D charged bosons in a magnetic field, i.e., the conditions of the quantum Hall effect. In this regime, it has been pointed out [7,8] that the celebrated Laughlin wave function is the exact ground state for the filling fraction $\nu = 1/2$, where ν is the ratio of the number of bosons to the number of vortices. Some of the excitations above this ground state are quasiparticles with fractional statistics which may eventually be probed by laser manipulations [9]. Investigations by exact diagonalization have given evidence [5] for even more exotic states of matter [10,11], some involving parafermionic wave functions introduced in the context of the fractional quantum Hall effect for fermions [12].

In this Letter we investigate the quantum Hall states of bosons as a function of the filling ν by use of exact diagonalizations in the spherical geometry [13,14]. This allows one to separate bulk from edge excitations. We show the appearance of the Bose analog of the Jain principal sequence of fractions, $\nu = \frac{n}{n+1}, \frac{p}{p-1}$. The excited states show collective modes consistent with a composite fermion picture in which there is binding of one vortex per boson. We obtain evidence for the Pfaffian state [10,11] at $\nu = 1$ by displaying its peculiar half-vortex excitations. We give gap estimates for $\nu = 1, 2/3, 1/2$ by a study of charged and neutral excitations. For higher fillings, $\nu \gtrsim 3/2$, we observe some states with properties of the Read-Rezayi parafermionic states. However, they

show no clear tendency to convergence to the thermodynamic limit.

In the rotating frame [15], the Hamiltonian describing N bosons of mass m is given by

$$
\mathcal{H} = \sum_{i=1}^{N} \frac{1}{2m}(\mathbf{p}_i - m\omega\hat{\mathbf{z}} \times \mathbf{r}_i)^2 \\
+ \frac{1}{2}m(\omega_0^2 - \omega^2)(x_i^2 + y_i^2) + \frac{1}{2}m\omega_z^2 z_i^2 \\
+ \sum_{i<j}^{N} V(\mathbf{r}_i - \mathbf{r}_j), \tag{1}
$$

where the xy trap frequency is ω_0, the axial frequency is ω_z, and the angular velocity vector is $\omega\hat{\mathbf{z}}$. In the ultracold atomic gases the interaction takes place through s-wave scattering only and is thus given by $V(\mathbf{r}) = (4\pi\hbar^2 a_s/m)\delta^{(3)}(\mathbf{r})$ where a_s is the s-wave scattering length. For ω close to ω_0, the physics is that of charge-e bosons in a magnetic field $\mathbf{B} = (2m\omega/e)\hat{\mathbf{z}}$, corresponding to a magnetic length $\ell = \sqrt{\hbar/2m\omega}$. There is then a 2D regime in which the boson wave function along the z axis is the ground state of the harmonic oscillator and the interaction is now given by $V^{2D}(\mathbf{r}) = g\ell^2\delta^{(2)}(\mathbf{r})$ (the vector $\mathbf{r}$ is 2D), with $g = \sqrt{32\pi}\hbar\omega a_s/\ell_z$, $\ell_z = \sqrt{\hbar/m\omega_z}$ is the confinement length along z. The energy scale of the quantum Hall problem is thus set by g.

We are thus led to study the quantum Hall effect of bosons interacting through a delta potential in the lowest Landau level (LL) [16]. To study the vortex liquids that appear as a function of the filling factor ν, we use the spherical geometry [14,17] in which the bosons move on a sphere of radius R in the magnetic field of a monopole $B = \hbar S/eR^2$ at the center of the sphere, giving rise to $2S + 1$ cyclotron orbits in the lowest LL. In the thermodynamic limit, the filling factor ν is given by $N/2S$. However, for the incompressible liquids there is, in general, a finite *shift* in the relation between the number of particles and the flux, i.e., one has generally $2S = (1/\nu)N - X$. If we have a guess for the ground state then one can evaluate the shift and check for its validity

against numerical results. For example, the bosonic Laughlin state for $\nu = 1/2$ on the sphere is realized for $2S = 2N - 2$ by the wave function

$$\Psi_{1/2} = \prod_{i<j}(u_i v_j - u_j v_i)^2, \qquad (2)$$

where the spinor coordinates (u, v) are given by

$$(u_i, v_i) = (\cos\theta_i/2 e^{i\phi_i/2}, \sin\theta_i/2 e^{-i\phi_i/2}). \qquad (3)$$

This is an exact zero-energy eigenstate of the present problem [8]. We have conducted Lanczos diagonalizations for various N and flux $2S$ to elucidate the nature of the incompressible liquid states. States can be labeled by their total angular momentum L, contrary to the planar geometry where only the z component is conserved.

Jain sequence.—The signature of incompressible states is the presence of a $L = 0$ singlet ground state separated by a clear gap from excited states. A typical spectrum is given in Fig. 1(a) for $\nu = 1/2$. In the excited states we observe a well-defined collective mode which is gapped for all values of L. Signs of incompressible liquids are also seen for the fractions $\nu = 2/3, 3/4, 4/3, 5/4$; some sizes are displayed in Figs. 1(b)–1(d). These fractions are the bosonic analog of the Jain sequence [18] of fractional quantum Hall states. This sequence was first reported in the disk geometry [5]. Such fractions can be explained by a composite particle picture in which the composite fermion (CF) is a boson bound to one vortex. Then the integer quantum Hall effect with n filled CF LLs leads to a fraction at $\nu = \frac{n}{n+1}$. This state is realized on the sphere for $2S = (n + 1)N/n - (n + 1)$, which is exactly what we observe. The collective mode is then an excitonlike mode obtained by promoting one CF from the highest occupied LL to the next LL. On the sphere the maximal L is then given by $L_{max} = (N/n) + n - 1$, in complete agreement with our results. The series of values for $(N, 2S)$ is *aliased* [19] with the sequence of fractions at

$\nu = \frac{p}{p-1}$. Indeed, if $(N, 2S)$ matches the fraction $n/(n + 1)$ then it also matches the fraction $p/(p - 1)$ for $p = N/n$. Hence, the same data set points to the presence of the fractions $\nu = 4/3$ and $5/4$ (we do not have enough points to provide gap estimates). The two-particle correlation function $g(r)$ is displayed in Fig. 2 for some states in the CF sequence. For $\nu = 1/2$ it is essentially free of finite-size effects and shows the strong correlation hole characteristic of a Laughlin ground state. The state with $\nu = 2/3$ no longer vanishes at the origin and hence has a nonzero ground state energy. The CF sequence also include the fractions $\nu = 3/2$ and $\nu = 2$. For these values we find families of incompressible states in the $(N, 2S)$ plane but they show no sign of convergence toward the thermodynamic limit, neither in the ground state energies nor in the gap values. For these fractions, we have candidates possibly originating from the Read-Rezayi parafermionic wave functions (see below). The gap to neutral excitations can be estimated by finite-size scaling. Our data for $\nu = 1/2$ are plotted versus $1/N$ in Fig. 3; there is smooth convergence towards ≈ 0.09. This gap can also be estimated from the energies of the Laughlin quasielectron and quasihole. For $\nu = 1/2$, the quasiholes have exactly zero energy since the interaction is a delta function. The quasielectron gap versus $1/N$ is displayed in Fig. 3; it converges towards a value close to that of the neutral excitation. We quote our final result as 0.095(5), the error being estimated from the dispersion of the values when we vary the fitting procedure as well as the difference between the two methods (charged [20] versus neutral excitations). For $\nu = 2/3$, both charged excitations are gapped. Their energies and those of the neutral mode are displayed in Fig. 3. We estimate the gap in this case as 0.05(1). Even if the N values are the same for $1/2$ and $2/3$, the flux per particle is closer to the thermodynamic limit for $1/2$ than for $2/3$. This is more and more problematic as one goes down the hierarchy.

For fillings less than $1/2$, the spectrum has many zero-energy eigenstates that are the quasiholes of the Laughlin state Eq. (2). This obscures the appearance of fractions
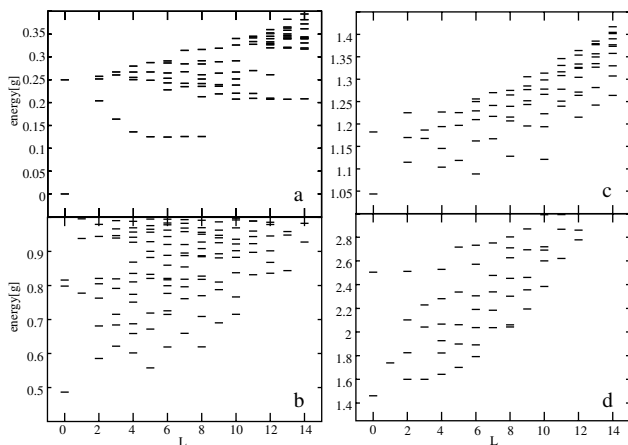

FIG. 1. Energy spectrum for (a) $\nu = 1/2$ ($N = 8, 2S = 14$), (b) $\nu = 2/3$ ($N = 8, 2S = 9$), (c) $\nu = 3/4$ ($N = 12, 2S = 12$), (d) $\nu = 4/5$ ($N = 8, 2S = 5$). Energies are in units of g and the horizontal axis is total angular momentum.

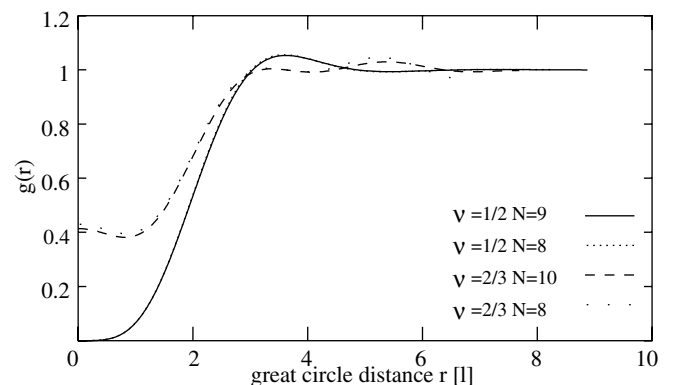FIG. 2. Two-particle correlation function $g(r)$ as a function of great circle distance in units of ℓ. The $\nu = 1/2$ curve is plotted for sizes $N = 8, 9$ and for $2/3$ $N = 8, 10$.
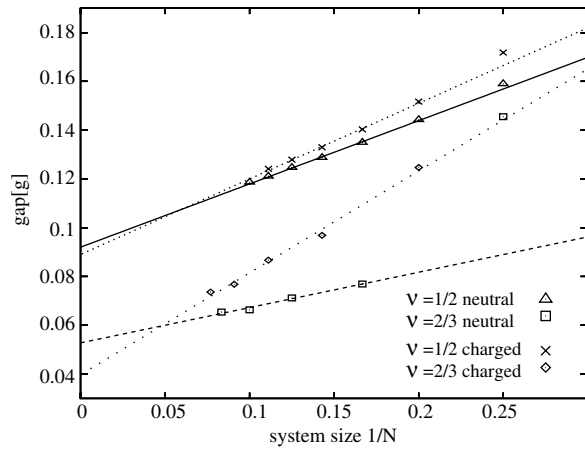
FIG. 3.   Gaps for the Bose-Jain sequence $\nu = 1/2$ and $2/3$. The lines are linear fits.

FIG. 4.   (a) Spectrum at the Pfaffian matching condition $N = 12$, $2S = 10$, with a collective mode above an isolated singlet ground state (b) with one extra flux quantum, two quasiparticles give rise to a degenerate set of states $L = 0, 2, 4, 6$.

less than $1/2$. If we change the interaction from pure delta by adding a pseudopotential V_2 [14] in the next allowed partial wave for bosons, i.e., $\ell = 2$, then we find other states from the hierarchy, the strongest being $\nu = 2/5$.

Pfaffian state.—The filling $\nu = 1$ corresponds to the absence of magnetic field acting upon the composite fermions. Previous studies [21] are indicative that pairing of the CFs takes place instead of a Fermi surface. An appealing wave function describing this phenomenon is the so-called Pfaffian state. On the sphere it can be written as

$$\Psi_{\nu=1} = \text{Pf}\left\{\frac{1}{u_i v_j - u_j v_i}\right\}\prod_{i<j}(u_i v_j - u_j v_i), \qquad (4)$$

where Pf stands for the Pfaffian, which is the antisymmetrized product of pair wave functions [22](a fermionic version of this state is a good candidate to describe the enigmatic $\nu = 5/2$ quantum Hall state). Calculations of overlaps between the model Pfaffian wave function Eq. (4) and the exact ground state suggest that it describes the physics of bosons at $\nu = 1$ in toroidal and disk geometry [5,7,8]. The bosonic Pfaffian state is realized on the sphere for $2S = N - 2$ for all N even. Our calculations lead to incompressible states at these special values for $N = 4$–16. There is a gap to a neutral excitation branch that appears in Fig. 4 for $N = 12$. The scaling of this gap for N ranging from 4 to 16 is displayed in Fig. 5. This state also has charged excitations that are different from those of a Laughlin fluid. If we add or remove one flux quantum, then *two* quasiparticles are created, leading to a set of low-lying states with an alternate even-odd character; see Fig. 4(b). This is consistent with the spectrum for two identical particles with repulsive interactions. This is observed for all accessible sizes. We consider this as a proof that the physics is different from that of the CF sequence and is the hallmark of the Pfaffian state [11]. The gap calculated from charged excitations does not scale as well as the neutral gap; see Fig. 5. This is clearly different from the case of fermions [19]. Our estimate of
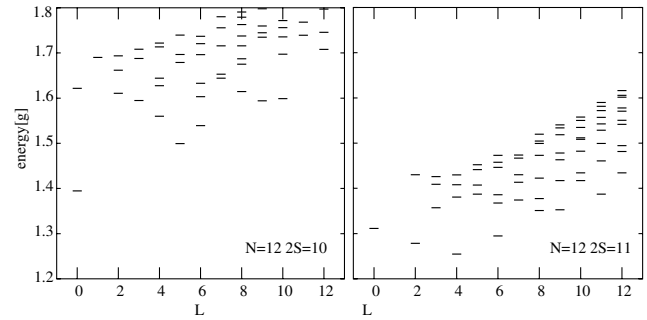
the Pfaffian gap is $0.05(1)$. The correlation function $g(r)$ is shown in Fig. 6. It now has a hump at the origin possibly due to the pairing of the CFs. The energy of this state is lowered by a correlation dip (instead of a hole) which appears at some characteristic radius of the order of 1 magnetic length. For larger separation, $g(r)$ approaches 1 but with a characteristic length scale which is definitely larger than that occurring in Fig. 2. The correlation length of the Pfaffian is larger than for the Jain-like fractions.

To investigate the stability of the Pfaffian state, we studied a potential with two partial waves, $L = 0$ and 2. In the range $|V_2/V_0| \lesssim +0.25$ we find that the Pfaffian state remains the ground state. It is replaced by a compressible liquid when we go outside this window. We have also studied a one-parameter potential interpolating from the delta interaction to the Coulomb interaction by adding the Coulomb pseudopotentials multiplied by a common factor. We find that the Pfaffian evolves smoothly up to the Coulomb point. The gap extrapolates always to a nonzero value, we also observe a change of
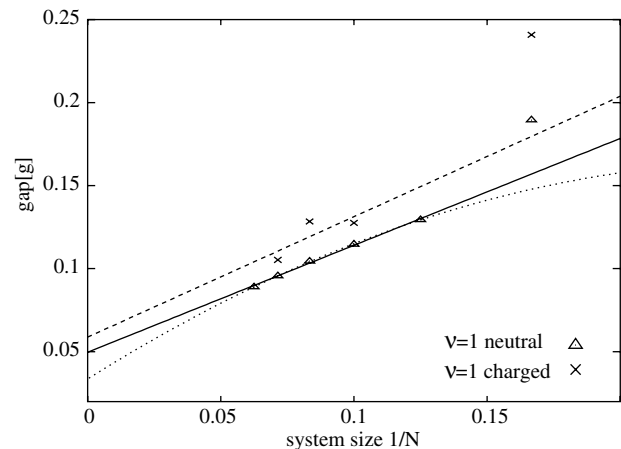

FIG. 5.   Gaps for the Pfaffian at $\nu = 1$. The dashed line is a linear fit to the charged excitations of the three largest systems, the solid (respectively, dotted) line is a linear (respectively, quadratic) fit of the neutral excitations.
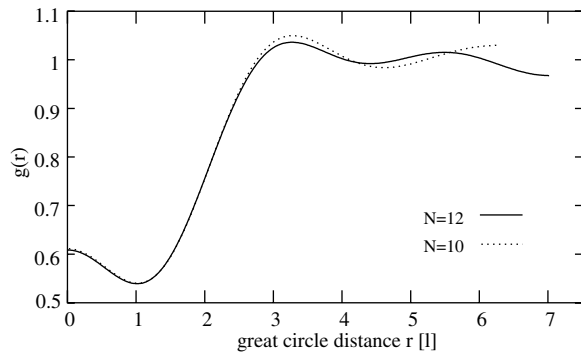
FIG. 6. The two-particle correlation function for the exact ground state at the Pfaffian point $2S = N - 2$. It has an extra concentration of bosons at the origin.

the finite-size effects: the curvature of the neutral mode versus $1/N$ becomes upwards for long-range interactions.

Read-Rezayi states.—For larger fillings it has been suggested by Cooper *et al.* [5] that fractions occur at $\nu = k/2$ and are well described by the Read-Rezayi parafermionic wave functions. These functions involve clustering of k particles and are a generalization of the Pfaffian which corresponds to simple pairing, i.e., $k = 2$. On the sphere they are realized for $2S = \frac{2}{k}N - 2$. There are possible candidates at $\nu = 3/2$ for $N = 6, 9, 12, 15$, $\nu = 2$ for $N = 8, 12, 16$, and the fraction $\nu = 5/2$ may be realized for $N = 15$ and 20 (but the gap is very small). We have computed the gaps to neutral and charged excitations: they do not show clear signs of convergence towards a thermodynamic limit. The gap is nonmonotonous as a function of the size for $\nu = 3/2$, and for $\nu = 2$ finite-size effects are very large, preventing any sensible extrapolation. One possibility is that these states have very large correlation lengths and are not accommodated on our largest spheres. This is consistent with the fact that the correlation function shows very strong oscillations and no hint of incompressibility. Evidence for clustering of more particles comes from the correlation function where we see the same phenomenon as in the Pfaffian case. The hump at the origin is even more pronounced and surrounded by a correlation hole. The hump also increases with the filling albeit we cannot make quantitative statements.

If we increase the number of bosons at fixed flux, then the spectrum becomes rotorlike: the levels lie on parabolas described by effective Hamiltonian $\frac{1}{2I}\mathbf{L}^2$ and correlation effects disappear.

We have shown the appearance of the Jain principal sequence of quantum Hall fractions in rotating Bose-Einstein condensates. The composite fermion picture gives a successful account of the observed fractions as well as their collective mode excitations. The Pfaffian state is realized at $\nu = 1$, as seen from the special matching of flux and number of bosons as well as its half-flux

quasiparticle. The gaps we estimate from our diagonalizations are all of the order of $\hbar \omega a_s / \ell_z$.

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