Does a Single Zealot Affect an Infinite Group of Voters?

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A method for studying the exact properties of a class of *inhomogeneous* stochastic many-body systems is developed and presented in the framework of a voter model perturbed by the presence of a "zealot," an individual allowed to favor an "opinion." We compute exactly the magnetization of this model and find that in one (1D) and two dimensions (2D) it evolves, algebraically ($\sim t^{-1/2}$) in 1D and much slower ($\sim 1/\ln t$) in 2D, towards the unanimity state chosen by the zealot. In higher dimensions the stationary magnetization is no longer uniform: the zealot cannot influence all the individuals. The implications to other physical problems are also pointed out.

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Researchers have devoted much attention to the field of nonequilibrium many-body stochastic processes [1]. In particular, the study of exact solutions of prototypical models such as the *voter model* [2] has proven to be fruitful for understanding a large class of nonequilibrium phenomena [1]. However, more realistic systems taking into account the important dynamical effects of inhomogeneities, constraints, and disorder (see, e.g., [1,3] and references therein) are less understood. To gain comprehension of these situations, exact results for systems modeling inhomogeneous situations in any dimension are desirable, but rather scarce [3].

In this work, motivated by these considerations, and with the voter model as a paradigm, we present techniques for computing the exact properties (in any dimension) of a class of stochastic many-body systems with *inhomogeneities.*

The voter model is an Ising-like model where an "individual" (or spin) associated with a lattice site $\mathbf{r}$ can have two different opinions $\sigma_{\mathbf{r}} = \pm 1$ [2]. The dynamics of the system is implemented by randomly chosing one individual and assigning to it the value of the spin of one of its randomly chosen nearest neigbors. In the voter model, the global magnetization is conserved and the dynamics is Z_2 symmetric (invariance under the global inversion $\sigma_{\mathbf{r}} \rightarrow -\sigma_{\mathbf{r}}$). The importance of the voter model stems from the fact that it is one of a very few stochastic many-body systems that are solvable in any dimension and is useful for describing the kinetics of catalytic reactions [4–6], in studying coarsening phenomena [6,7] and as a prototype model of opinion dynamics [8].

For the sake of concreteness, and without loss of generality, we specifically present our method and techniques in the framework of an *inhomogeneous voter model* where, to mimic in a simple manner the fact that a group of agents may have heterogeneous interactions, the conventional voter dynamics is supplemented by the presence of a "zealot": a biased individual who favors one opinion. We study the effect of this perturbation by computing the exact long-time magnetization, which is no longer conserved, in dimensions $d = 1, 2$, and 3. In low dimensions the zealot drastically affects the dynamics: the system evolves towards unanimity with the latter. The approach to the stationary state is algebraic in 1D and logarithmically slow in 2D. In $d \geq 3$, the effect of the zealot is less dramatic and the local stationary magnetization is a nontrivial function of the distance to this biased individual. These findings, although formulated in an "opinion dynamics" language, are, as pointed out hereafter, relevant to other physical problems.

The inhomogeneous voter model that we study is defined on a hypercubic lattice of size $(2L+1)^d$, where individuals, labeled by a vector $\mathbf{r}$ having components $-L \leq r_i \leq L$ (with $i = 1, \dots, d$), may interact according to the usual voter dynamics. In addition, we now consider that a zealot, at site "$\mathbf{0}$", tends to favor the diffusion of the $+1$ opinion via the interaction with his neighbors: the zealot is the only individual in the system allowed to change his state from -1 to $+1$ (with rate $\gamma > 0$) without regard to his neighbors, with whom he nevertheless interacts. According to the spin formulation of the model, the state of the system is described by the collection of all spins: $S \equiv \{\sigma_{\mathbf{r}}\}$. In this language, the dynamics of the model is governed by the usual voter model transition rate [1,2,4,5] supplemented by a local term involving the local zealot's reaction. The spin-flip rate, $w_{\mathbf{r}}(S) \equiv w(\sigma_{\mathbf{r}} \rightarrow -\sigma_{\mathbf{r}})$, therefore reads

$$w_{\mathbf{r}}(S) = \frac{1}{\tau}\left(1 - \frac{1}{2d}\sigma_{\mathbf{r}}\sum_{\mathbf{r}'}\sigma_{\mathbf{r}'}\right) + \frac{\gamma}{2}(1-\sigma_{\mathbf{0}})\delta_{\mathbf{r},\mathbf{0}}. \quad (1)$$

Here the sum on the right-hand side (rhs) runs over the $2d$ nearest neighbors $\mathbf{r}'$ of site $\mathbf{r}$ and $\tau \equiv 1/\beta d > 0$ defines the time scale. The probability distribution $P(S,t)$ satisfies the master equation:

$$\frac{d}{dt}P(S,t) = \sum_{\mathbf{r}}[w_{\mathbf{r}}(S^{\mathbf{r}})P(S^{\mathbf{r}},t) - w_{\mathbf{r}}(S)P(S,t)], \quad (2)$$

where the state $S^{\mathbf{r}}$ differs from S only by the spin-flip of $\sigma_{\mathbf{r}}$. With the master Eq. (2), in the limit $L \to \infty$, the equation of motion of the local magnetization $S_{\mathbf{r}}(t) \equiv \sum_S \sigma_{\mathbf{r}} P(S,t)$ reads

$$\frac{dS_{\mathbf{r}}(t)}{d(\beta t)} = \Delta_{\mathbf{r}} S_{\mathbf{r}}(t) - \frac{\gamma}{\beta}[S_{\mathbf{0}}(t) - 1]\delta_{\mathbf{r},\mathbf{0}}. \quad (3)$$

Here $\Delta_{\mathbf{r}}$ denotes the discrete Laplace operator: $\Delta_{\mathbf{r}} S_{\mathbf{r}}(t) \equiv -2dS_{\mathbf{r}}(t) + \sum_{\mathbf{r}'} S_{\mathbf{r}'}(t)$.

The last term on the rhs of Eq. (3) is due to the effect of the zealot and appears only in the case where $\mathbf{r} = \mathbf{0}$.

An important consequence of Eq. (3) is that the only possible *uniform* final magnetization is $S_{\mathbf{r}}(\infty) = 1$, i.e., the state favored by the zealot. However, in $d \geq 3$, as shown hereafter, the stationary magnetization profile turns out to be *nonuniform* (but isotropic).

Using the properties of the modified Bessel functions of the first kind, $I_r(t)$ [9], we obtain the formal solution of Eq. (3):

$$S_{\mathbf{r}}(t) = \sum_{\mathbf{k}} S_{\mathbf{k}}(0) \prod_{i=1}^{d} [e^{-2\beta t} I_{k_i - r_i}(2\beta t)] + \gamma \int_0^t dt' [1 - S_{\mathbf{0}}(t-t')] \prod_{i=1}^{d} [e^{-2\beta t'} I_{r_i}(2\beta t')]. \quad (4)$$

To obtain an explicit expression for the magnetization we solve the self-consistent integral Eq. (4) for $\mathbf{r} = \mathbf{0}$ and then plug the result back into (4). For this purpose it is useful to denote the Laplace transform of a product of Bessel functions (multiplied by an exponential term)

$$\hat{I}_{\mathbf{r}}(s,\beta) \equiv \int_0^\infty dt\, e^{-st}[e^{-2d\beta t} I_{r_1}(2\beta t) \ldots I_{r_d}(2\beta t)]. \quad (5)$$

This quantity can be rewritten in terms of Watson integrals, or "lattice Green-functions":

$$\hat{I}_{\mathbf{r}}(s,\beta) \equiv \int_{-\pi}^{\pi} \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{e^{-i\mathbf{q}.\mathbf{r}}}{s + 2\beta\left[d - \sum_{i=1}^{d} \cos q_i\right]}, \quad (6)$$

where $\mathbf{q} = (q_1, \ldots, q_d)$ is a d-dimensional vector.

Laplace-transforming Eq. (4), and using the convolution theorem, we obtain the following expression for the Laplace transform of the local magnetization:

$$\hat{S}_{\mathbf{r}}(s) \equiv \int_0^\infty dt\, e^{-st} S_{\mathbf{r}}(t) = \frac{\gamma \hat{I}_{\mathbf{r}}(s,\beta)}{s[1 + \gamma \hat{I}_{\mathbf{0}}(s,\beta)]}, \quad (7)$$

where, for technical simplicity, we consider the zero initial magnetization state: $S_{\mathbf{k}}(0) = 0$.

The exact expression for the long-time magnetization is obtained by Laplace inverting the $s \to 0$ expansion of Eq. (7) and paying due attention to the situations where the integrals (6) are divergent. In the sequel the norm of the vector $\mathbf{r}$ is denoted r.

We first consider the one-dimensional problem. In this case, the quantity (5) reads [9]:

$$\hat{I}_{\mathbf{r}}(s,\beta) \equiv \hat{I}_r(s,\beta) = \frac{\{[\sqrt{s+4\beta} - \sqrt{s}]/(2\sqrt{\beta})\}^{2r}}{\sqrt{s(s+4\beta)}}. \quad (8)$$

It should be noticed that the $s \to 0$ behavior of (8) diverges as $s^{-1/2}$.

Laplace inverting the $s \to 0$ expansion of (7), together with the expression of (8) for $r = 0$, we get ($\beta t \to \infty$):

$$S_0(t) = 1 - \frac{2}{\gamma}\sqrt{\frac{\beta}{\pi t}}. \quad (9)$$

In the more general case, where $r > 0$, the local magnetization is obtained similarly from (7) and (8) [in the limit $s \to 0$, with $r\sqrt{s}$ kept fixed]:

$$S_r(t) \simeq \mathrm{erfc}\left(\frac{r}{2\sqrt{\beta t}}\right), \quad (10)$$

where $\mathrm{erfc}(z)$ is the usual complementary error function [9]. The expression (10) is valid for $\beta t \gg 1$, and is particularly useful in the scaling limit where both $r \to \infty$ and $t \to \infty$, but where the ratio $r/\sqrt{\beta t}$ is finite.

When r is finite (i.e., $0 < r < \infty$) and $\beta t \to \infty$, we obtain the following long-time behavior:

$$S_r(t) = 1 - \frac{r + (2\beta/\gamma)}{\sqrt{\pi \beta t}}. \quad (11)$$

In two dimensions, the integral (6) is also divergent in the long-time regime $s \to 0$ and therefore its main contribution arises from $q^2 \equiv q_1^2 + q_2^2 \to 0$. In this sense, we first expand (6) for small s in the case where $\mathbf{r} = \mathbf{0}$:

$$\hat{I}_{\mathbf{0}}(s,\beta) \underset{s\to 0}{\longrightarrow} -\frac{1}{4\pi\beta}\ln\left[\frac{s}{\beta}\right], \quad (12)$$

which implies that the long-time behavior of the average-opinion of the zealot is

$$S_{\mathbf{0}}(t) - S_{\mathbf{0}}(\infty) \simeq -\left(\frac{4\pi\beta}{\gamma}\right)\frac{1}{\ln \beta t}, \quad (13)$$

where $S_{\mathbf{0}}(\infty) = 1$.

For the other individuals we proceed similarly and from (6), with $r \gg 1$, we have

$$\hat{I}_{\mathbf{r}}(s,\beta) \underset{s\to 0}{\longrightarrow} \frac{1}{2\pi\beta} K_0\left(r\sqrt{\frac{s}{\beta}}\right), \quad (14)$$

where $K_0(x)$ is the usual modified Bessel function of the third kind [9]. Using the small argument expansion of such a Bessel function [9], we find that the long-time behavior in the region where $\beta t \gg r^2 \gg 1$ is

$$S_{\mathbf{r}}(t) - S_{\mathbf{r}}(\infty) \sim -\frac{\ln r^2}{\ln \beta t}, \quad (15)$$

where the stationary magnetization corresponds again to the unanimous opinion, as in 1D: $S_{\mathbf{r}}(\infty) = 1$.

The regime where $r^2 \propto \beta t \to \infty$ should still be discussed separately. In fact, in this regime, from (7), (12), and (14):

$$\hat{S}_{\mathbf{r}}(s) \underset{s\to 0, r\to\infty}{\longrightarrow} -\frac{2K_0(r\sqrt{s/\beta})}{s\ln(s/\beta)}, \qquad (16)$$

where $s \to 0$ and $r \to \infty$, but $r\sqrt{s/\beta}$ is finite.

From results (10), (11), (15), and (16) we infer that in low dimensions, at large time, the effect of the zealot appears at two length scales: (i) The opinion of individuals "close" ($r^2 \ll \beta t$) to the zealot evolves algebraically ($\propto t^{-1/2}$) in 1D and logarithmically ($\propto 1/\ln t$) in 2D towards the unanimous opinion $S_{\mathbf{r}}(\infty) = 1$. (ii) For individuals "far" ($r^2 \propto \beta t \to \infty$) from the zealot, the local magnetization evolves as a smooth scaling function of $u \equiv r^2/2\sqrt{\beta t}$ in 1D. This is, however, no longer the case in 2D, where, due to logarithmic terms arising in (16), the magnetization has not a scaling form. A qualitatively similar result, but within a different context, has recently been reported in Ref. [10]. One can also notice that results (10), (11), (15), and (16) are (mainly) independent of the rate γ: for the long-time behavior of the local magnetization; in low dimensions, only the fact that there is a biased individual (i.e., $\gamma > 0$) matters.

We therefore conclude that both in 1D and 2D the zealot eventually affects all the individuals: the number of voters having a final $+1$ opinion are within a "circle" whose radius increases as $\sqrt{\beta t} \to \infty$.

We now study the three-dimensional situation, and then consider its generalization to the case where $d \geq 3$.

When $d \geq 3$, the integrals (6) are well defined for all the values of s, and, in particular, when $s \to 0$. Therefore, conversely to what happens in 1D and 2D, to determine the long-time behavior of the magnetization we cannot simply focus on the $q \to 0$ expansion of (6). Fortunately, very recently Glasser and Boersma have been able to explicitly compute (6) in the 3D case where $s = 0$ [11]. We now take advantage of these findings to compute the stationary magnetization in 3D. We therefore introduce a triplet $(a_{\mathbf{r}}, b_{\mathbf{r}}, c_{\mathbf{r}})$ of rational numbers depending on $\mathbf{r}$, given in Table II of Ref. [11], and the quantity $g_0 \equiv [(\sqrt{3}-1)/96\pi^3]\Gamma^2(\frac{1}{24})\Gamma^2(\frac{11}{24})$, where $\Gamma(z)$ is Euler's Gamma function [9].

With help of the results obtained in [11], we are in a position to compute the 3D stationary local magnetization (SLM) by taking the $s \to 0$ limit in (7):

$$S_{\mathbf{r}}(\infty) = \frac{\gamma[a_{\mathbf{r}}g_0^2 + c_{\mathbf{r}}g_0 + \frac{b_{\mathbf{r}}}{\pi^2}]}{g_0[2\beta + \gamma g_0]}. \qquad (17)$$

In particular, for $\mathbf{r} = \mathbf{0}$ we have the triplet $(a_{\mathbf{0}}, b_{\mathbf{0}}, c_{\mathbf{0}}) = (1, 0, 0)$ [11] and thus obtain $S_{\mathbf{0}}(\infty) = [\gamma g_0/(2\beta + \gamma g_0)]$.

To gain, in a simple manner, some further insight of the behavior of the discrete expression (17) when $r \to \infty$, it turns out to be fruitful to take the *continuum limit* of (3). In this limit the SLM is $S(\mathbf{r}, \infty)$, and we have to solve the problem of determining the electric potential due to a "charge" at the origin. One should pay due attention to the fact that this electrostatic reformulation needs to be supplemented by additional information since, from the continuum limit of (3), the charge is *a priori* an unknown quantity. To overcome this difficulty, in 3D, we take advantage of our knowledge of the discrete version of the problem and, with (17), we compute the charge at site $\mathbf{r} = \mathbf{0}$ as $\frac{\gamma}{\beta}[1 - S_{\mathbf{0}}(\infty)] = [2\gamma/(2\beta + \gamma g_0)]$. We therefore obtain the following 3D continuum stationary equation:

$$\Delta S(\mathbf{r}, \infty) = -\frac{2\gamma}{2\beta + \gamma g_0}\,\delta^3(\mathbf{r}), \qquad (18)$$

where Δ is the 3D-Laplacian operator and $\delta^3(\mathbf{r})$ denotes the 3D-Dirac delta function.

The solution of Eq. (18) depends on γ and reads:

$$S(\mathbf{r}, \infty) = \frac{\gamma}{2\pi(2\beta + \gamma g_0)}\,\frac{1}{r}, \qquad (r > 0). \qquad (19)$$

Comparing the predictions of the results (17) and (19), we notice that they agree very well, even for finite r: For instance, when $\gamma = 2\beta$, at $\mathbf{r} = (3, 1, 0)$, we have $S_{\mathbf{r}}(\infty) = 0.0339$ and $S(\mathbf{r}, \infty) = 0.0334$, whereas for $\mathbf{r} = (3, 1, 1)$ $S_{\mathbf{r}}(\infty) = 0.0319$ and $S(\mathbf{r}, \infty) = 0.0319$.

Results (17) and (19) show that in 3D (conversely to what happens in low dimensions) the SLM is an isotropic [which is clear from (6) and (19)] but *nonuniform function* decaying with the norm of $\mathbf{r}$: $S_{\mathbf{r}}(\infty) \approx S(\mathbf{r}, \infty) = [A_3(\gamma)/r]$, where the amplitude is given by (19) [and $r > 0$]. The reasoning can be extended to dimensions $d > 3$, where the electrostatic reformulation gives the result (for $r > 0$): $S(\mathbf{r}, \infty) = (A_d/r^{d-2})$. Again the computation of the amplitude A_d requires the explicit knowledge of $\hat{I}_{\mathbf{0}}(s = 0, \beta)$ in dimensions $d > 3$.

Despite the fact that in $d \geq 3$ the $s \to 0$ analysis of (6) is a difficult task, we can infer (for r finite) the long-time behavior from (4): $S_{\mathbf{r}}(t) - S_{\mathbf{r}}(\infty) \sim -(\beta t)^{1-(d/2)}$.

From Eq. (4) we can also compute the total magnetization: $M(t) \equiv \sum_{\mathbf{r}} S_{\mathbf{r}}(t) = M(0) + \gamma \int_0^t d\tau[1 - S_{\mathbf{0}}(\tau)]$. This expression shows that in the voter model ($\gamma = 0$) the quantity $M(t)$ is conserved, which is no longer the case when $\gamma > 0$. With the help of the results (9), (13), and (17), we obtain the long-time behavior of $M(t)$ [here $M(0) = 0$]:

$$M(t) \sim \begin{cases} (\beta t)^{1/2}, & d = 1, \\ \beta t/\ln t, & d = 2, \\ \beta t, & d = 3. \end{cases} \qquad (\beta t \to \infty). \qquad (20)$$

These results show that in this inhomogeneous voter model the saturation time t_s, that is, the time necessary to have $M(t_s)$ comparable to the size L^d (where $L \to \infty$) of the system, scales as $\beta t_s \sim L^2$ in 1D, $\beta t_s \sim L^2 \ln L$ in 2D, and $\beta t_s \sim L^d$ for $d \geq 3$. These statements are in agreement with results obtained for other models (see, e.g., [4]).

In this work we have developed a method to compute some exact properties of a class of stochastic many-body problems with inhomogeneities and have explicitly presented this approach in the framework of an inhomogeneous voter model where the usual voter dynamics is perturbed by the local presence of a single zealot. For this opinion dynamics problem, we have computed exactly, in dimensions $d = 1, 2$, and 3, the long-time magnetization (mean-average opinion of each voter). From our exact results we have seen that in low dimensions the zealot (i.e., the inhomogeneity) always affects the mean-average opinion and that its effect propagates as $t^{1/2}$. In fact the mean opinion of individuals approaches, algebraically in 1D [see (11)], according to the scaling expression (10), and logarithmically slowly in 2D [see (15)], the unanimity state favored by the zealot. These results are (mainly) independent of the strength of the biased individual. In 3D the situation is completely different and the (stable) stationary mean-average opinion of a voter is no longer uniform but follows the nontrivial isotropic function (17) that decays with the inverse of the distance to the zealot [see (19)], a result also obtained in the continuum limit via a suitable electrostatic reformulation, and then extended to the case where $d > 3$. The findings obtained in this work, and the differences between the behaviors observed in low dimensions and in $d \geq 3$, can be qualitatively understood in realizing that the local magnetization is the solution of a diffusionlike Eq. (3) supplemented by a local boundary term (the zealot) and taking into account the fact that in 1D and 2D random walks are recurrent (which, in our case, implies that all individuals interact with the zealot), while in $d \geq 3$ they are transient, and therefore there is a finite probability that individuals never interact with the zealot [1,2]. It is instructive to compare our results to those obtained in the conventional voter model [4–7]: the presence of the zealot clearly implies that the magnetization is no longer conserved and that the dynamics is not translationally invariant. This comparison also shows that a single inhomogeneity (here, the zealot) can deeply affect the stationary and the long-time properties of an interacting spin system, whose perturbed dynamical behavior depends on the dimension d.

Despite the fact that our method and results have been formulated in an opinion dynamics language, we emphasize that they have a broad physical relevance and can be applied to a large class of stochastic many-body problems. As a physical illustration we can consider the kinetics of the monomer-monomer catalysis on an inhomogeneous substrate that locally desorps preferentially one species of monomer [4,5,12]. Identifying spins $+1$ (-1) with A (B) particles, the monomer-monomer reaction can be mapped onto an Ising model with mixed voter and Kawasaki dynamics [4,5], whose time scales are, respectively, defined by $1/\beta d$ and $1/\beta' d$, and an inhomogeneous term [as in (1)] that mimics the *local* desorption (at $\mathbf{r} = \mathbf{0}$), with rate γ, of particles of species B [4,12]. For this model, with $\tilde{\beta} \equiv \beta + \beta'$, the local concentration $c_{\mathbf{r}}(t)$ of A particles obeys to $[d/(\tilde{\beta}dt)]c_{\mathbf{r}}(t) = \Delta_{\mathbf{r}} c_{\mathbf{r}}(t) - (\gamma/\tilde{\beta})[c_{\mathbf{0}}(t) - 1]\delta_{\mathbf{r},\mathbf{0}}$, whereas the concentration of B particles, at site $\mathbf{r}$, is $1 - c_{\mathbf{r}}(t)$ [12]. Comparing the equation for $c_{\mathbf{r}}$ with Eq. (3), it is clear that the concentration of particles in the catalysis problem can immediately be inferred from the above results for $S_{\mathbf{r}}(t)$ [12]. Another field of pertinence of this work, according to the well-known relation between the (1D) voter and Glauber-Ising models [4,5,13], is the area of inhomogeneous magnetism. We can also mention that our method, which can take into account the presence of many inhomogeneities and can deal with systems of more than two states per site, is of direct relevance for a large class of reaction-diffusion models [1]. In this context, a diffusive model of interacting particles where an homogeneous source is in competition with a local trap has been solved [12].

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