Three Types of Transitions to Phase Synchronization in Coupled Chaotic Oscillators

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We study the effect of noncoherence on the onset of phase synchronization of two coupled chaotic oscillators. Depending on the coherence properties of oscillations characterized by the phase diffusion, three types of transitions to phase synchronization are found. For phase-coherent attractors this transition occurs shortly after one of the zero Lyapunov exponents becomes negative. At rather strong phase diffusion, phase locking manifests a strong degree of generalized synchronization, and occurs only after one positive Lyapunov exponent becomes negative. For intermediate phase diffusion, phase synchronization sets in via an interior crises of the hyperchaotic set.

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Phase synchronization (PS) of chaotic oscillators [1] and rotators [2] has been detected in many natural [3] and engineering [4] systems. In spite of the fact there are a large number of publications in this field (for a review, see [5]), several important aspects have not been well understood. It is still unknown which type of chaotic oscillators can be synchronized in phase. How does the structure of the chaotic attractor influence the transition to PS? Until now chaotic phase synchronization (CPS) has been mainly observed for chaotic attractors with rather coherent phase dynamics. These attractors have a relatively simple topology of oscillations and a well-pronounced peak in the power spectrum, which allows the introduction of the phase and the characteristic frequency of motions. Thus, CPS in this case is rather similar to PS of periodic oscillations in the presence of small noise. The transition to PS is then accomplished by the transition of a zero Lyapunov exponent to negative values. However, such a phase-coherent dynamic is a strong restriction for applications. We often meet noncoherent attractors and rather broad band power spectra. Then a phase of the oscillations may not be defined straight forwardly, and there in general does not exist only one single characteristic time scale. In contrast to phase-coherent attractors, the analogy with noisy periodic oscillators is then no longer evident, and whether some phase synchronized state can be achieved is quite unclear.

To treat such systems, we propose in this Letter a method to more generally define a phase. This allows one to study CPS in systems of coupled chaotic oscillators with even strongly noncoherent phase properties. Based on this approach, we find three possible types of transition to CPS in dependence on the coherence properties of motions. This is demonstrated for the paradigmatic system of two coupled nonidentical Rössler oscillators:

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2}, \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2} + d(y_{2,1} - y_{1,2}), \\ \dot{z}_{1,2} &= 0.1 + z_{1,2}(x_{1,2} - 8.5), \end{aligned} \tag{1}$$

where d is the coupling strength. $\omega_{1,2}$ determine the mean frequency of the oscillators in the case of phase-coherent attractors. In our simulations we take $\omega_1 = 0.98$ and $\omega_2 = 1.02$. The parameter $a \in [0.15{:}0.3]$ governs the topology of the chaotic attractor. When a is below a critical value a_c ($a_c \approx 0.186$ for $\omega_1 = 0.98$ and $a_c \approx 0.195$ for $\omega_2 = 1.02$), the chaotic trajectories always cycle around the unstable fixed point $(x_0, y_0) \approx (0, 0)$ in the (x, y) subspace, i.e., $\max(y) > y_0$ [Fig. 1(a)]. In this case, the rotation angle $\phi = \arctan\frac{y}{x}$ can be defined as the phase, which increases almost uniformly; i.e., the oscillator has a coherent phase dynamic. Beyond the critical

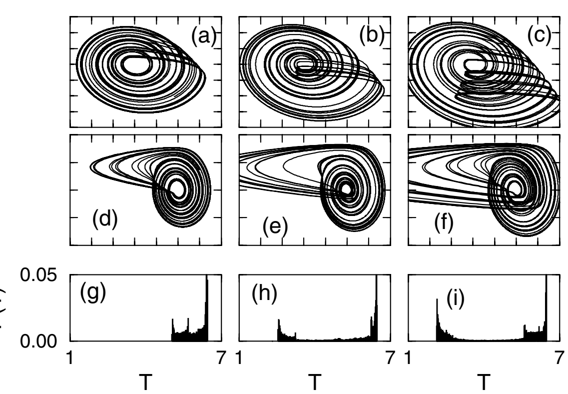

FIG. 1. Upper panel (a)–(c): projections of the attractors of the Rössler systems (1) onto the plane (x, y); middle panel (d)–(f): projections onto $(\dot{x}, \dot{y})$; lower panel (g)–(i): distribution of the return times T. The parameters are $\omega = 0.98$ and $a = 0.16$ (a),(d),(g), $a = 0.22$ (b),(e),(h), and $a = 0.28$ (c),(f),(i).

value a_c, the trajectories no longer always completely cycle around (x_0, y_0), and some $\max(y) < y_0$ occur, which are associated with faster returns of the orbits [Fig. 1(b)]. The attractor becomes a funnel attractor. Such earlier returns in the funnel attractor happen more frequently with increasing a [Fig. 1(c)]. For the funnel attractors, usual (and rather simple) definitions of phase [1] are no longer applicable.

We propose, therefore, another approach to define the phase based on the general idea of the curvature of an arbitrary curve [6]. For any two-dimensional curve $\vec{r}_1 = (u, v)$ the angle velocity at each point is $\nu = (ds/dt)/R$, where $ds/dt = \sqrt{\dot{u}^2 + \dot{v}^2}$ is the speed along the curve and $R = (\dot{u}^2 + \dot{v}^2)^{3/2}/[\dot{v}\,\ddot{u} - \ddot{v}\,\dot{u}]$ is the radius of the curvature. If $R > 0$ at each point, then $\nu = \frac{d\phi}{dt} = (\dot{v}\,\ddot{u} - \ddot{v}\,\dot{u})/(\dot{u}^2 + \dot{v}^2)$ is always positive and therefore the variable ϕ, defined as $\phi = \int \nu dt = \arctan(\dot{v}/\dot{u})$, is a monotonically growing angle function of time and can be considered as a phase of oscillations. Geometrically it means that the projection $\vec{r}_2 = (\dot{u}, \dot{v})$ is a curve cycling monotonically around a certain point.

The above definitions of phase and frequency are general for any dynamical system if the projection of the phase trajectory on some plane is a curve with a positive curvature. We find that they are applicable to a large variety of chaotic oscillators, such as Lorenz system [7], Chua circuit [8], Colpits oscillator [9], extended Bonhöffer-van der Pol oscillator [10], Anishchenko-Astakhov generator [11], and the model of the ideal four-level laser with periodic pump modulation [12].

For phase-coherent as well as funnel attractors in the Rössler oscillator, projections of chaotic trajectories on the plane $(\dot{x}, \dot{y})$ always rotate around the origin [Figs. 1(d)–1(f),] and the phase can be defined as [13]

$$\phi = \arctan\frac{\dot{y}}{\dot{x}}. \tag{2}$$

To see the influence of the noncoherence of the orbits on the time scales of the oscillations, we calculate the distribution of the return time T of the orbits between two successive crossings of the Poincaré section $\dot{y} = 0, \dot{x} > 0$. For the phase-coherent attractor (e.g., $a = 0.16$), T is located in a relatively narrow interval [Fig. 1(g)]. When the attractor becomes noncoherent, the earlier returns are associated with much smaller T, and the distribution becomes rather broad [Figs. 1(h) and 1(i)]. For intermediate a (e.g., $a = 0.22$) one characteristic time scale is still dominant, while for large a (e.g., $a = 0.28$) there exist two relatively well-pronounced time scales. As we will show below, this causes quite different transitions to CPS for the funnel attractors.

We measure the degree of noncoherence on time scales by *the phase diffusion coefficient* D_ϕ defined as

$$\langle(\phi(t) - \langle\phi(t)\rangle)^2\rangle = 2D_\phi t, \tag{3}$$

where $\langle\cdot\rangle$ denotes the ensemble average. D_ϕ increases in general with a interrupted by $D_\phi = 0$ in periodic windows. For a phase-coherent chaotic attractor, the phase increases approximately uniformly and D_ϕ is rather small. But for a funnel chaotic attractor the increase of phase is strongly nonuniform and D_ϕ can be a few orders larger in magnitude (Fig. 2).

We use two criteria to detect the existence of CPS [14]: locking of the mean frequencies $\Omega_1 = \langle\nu_1\rangle = \Omega_2 = \langle\nu_2\rangle$ and locking of phase $|\phi_2(t) - \phi_1(t)| \leq \text{const}$.

To illustrate the transition to CPS for different values of the parameter a, we inspect the change of the mean frequency ratio Ω_2/Ω_1 and the spectrum of Lyapunov exponents (LE) versus the coupling strength d.

Three types of qualitatively different transitions to CPS in dependence on a are clearly seen in the phase diagram (Fig. 3).

(i) In the well-studied case in the interval $a \in [0.15, 0.186]$, both oscillators have phase-coherent chaotic attractors. Here the zero LEs are associated with the phase dynamics. Because of the high degree of coherence of the motions, i.e., very small D_ϕ, phase and frequency locking occur shortly after the transition of one of the zero LEs to a negative value. The two largest LEs remain positive, and the amplitudes of the oscillators are only weakly correlated. A strong correlation of amplitudes sets in only at a much larger coupling ($d > d_{l_3}$), where one of the positive LEs becomes negative, and the two systems achieve generalized chaotic synchronization (GCS) [15].

(ii) A quite different scenario of transition to CPS takes place for $a \in [0.195, 0.25]$, where both chaotic attractors are funnel. Here the curves l_1 and l_2 are clearly separated, but both lie below curve l_3. The two largest LEs remain positive during the transition to CPS, which means that no bifurcation of the hyperchaotic attractor can be associated with the locking of the phases. CPS occurs here via a crisis transition inside the hyperchaos, i.e., via *an interior crises* of the chaotic set. Such a change of the internal structure of the attractor is seen by the projection of phase trajectory on the plane (ϕ_1, ϕ_2) for d outside [Fig. 4(a)] and within [Fig. 4(b)] the synchronized regime. In the case of a nonsynchronized state (but near the transition point), the plot covers practically the whole plane (ϕ_1, ϕ_2) with different density. The dense band corresponds to relatively long epochs of synchronization. The presence of a trajectory in the rest of the plane indicates the existence of phase slips. In the synchronized state the plot is restricted to the narrow bands.

(iii) In the interval $a \in [0.25, 0.3]$, curve l_1 lies above curve l_3, showing that CPS occurs after one of the positive LEs passes to negative values, i.e., the transition to

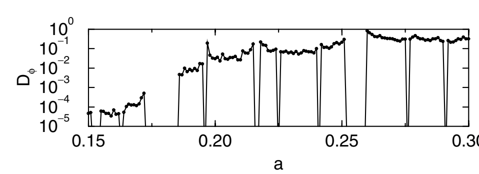

FIG. 2. Phase diffusion coefficient D_ϕ (3) vs a. $\omega = 0.98$.

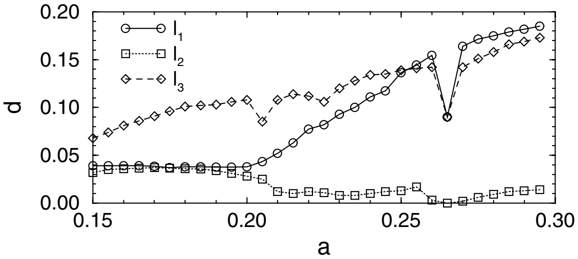

FIG. 3. Critical coupling curves. l_1 corresponds to the onset of CPS, i.e., below this line the oscillations are not synchronized, and above this line the phase and frequency locking conditions are fulfilled; l_2 corresponds to the transition of one of the zero LEs negative values; and l_3 corresponds to zero crossing of one of the positive LEs. Note that in this figure we do not separate the cases where the synchronization occurs between regular and chaotic oscillations.

GCS. It is important to note that, by GCS, both oscillators have established a rather strong cross correlation. However, such a strong relationship is an average property over the whole attractors, while locally a phase slip associated to a different number of oscillations in the two oscillators in a period of time may occur for the coupling strength d shortly above d_{l_3}, as seen in Fig. 5 for the typical behavior for $d \in [d_{l_3}, d_{l_1}]$. CPS appears as a manifestation of GCS. This property is in contrast to the above regimes where CPS is a weaker degree of synchronization compared to GCS, and the phases become locked before a strong correlation of the amplitudes can be established. Thus, for highly noncoherent oscillations due to the existence of two distinct characteristic time scales, a rather strong coupling is necessary to keep both oscillators in small or larger cycles simultaneously in order to maintain the phase locking. Otherwise, a phase slip develops quickly due to the very different time scales when the two oscillators are on a small and a large cycle, respectively. That is why phase locking becomes impossible without a strong correlation in the amplitudes.

To gain deeper insight into the transition routes to synchronized state, we study the scaling properties of intermittent phase slips near the CPS transition point. For a phase-coherent attractor very close to the transition point, phase slips occur very rarely and the average time interval τ between two successive slips is extremely long and can be described by $\tau \approx \exp(k|d - d_{l_1}|^{-1/2})$ [16]. Away from the critical point quite different scaling properties of the transitions to CPS are observed for phase-coherent and funnel attractors (Fig. 6). For phase-coherent attractors, due to the existence of only one well-pronounced time scale, the synchronization is achieved through a smooth frequency locking, and the behavior can be described by a type-I intermittency scaling law. While for the funnel attractors there exist small and large loops of the trajectory rotations. When the coupling is weak, it appears very often that the first oscillator is on the small loops while the second oscillator is on the large loops, or vice versa. This leads to an unpredictable, oscillatory-like evolution of the Ω_2/Ω_1 ratio. Relatively close to the critical point, rather strong coupling makes the oscillators stay simultaneously on small or large loops for a long period of time. The Ω_2/Ω_1 ratio decreases monotonically and can be fitted by a straight line in a large range below the critical point.

The difference in the transitions to CPS may be understood as follows. In [17] it was shown that CPS takes place in a parameter region where all unstable periodic orbits (UPO), embedded in the chaotic attractors, are synchronized. For phase-coherent attractors, periods of all UPOs are close to each other [see Fig. 1(g)] and the boundaries of the Arnold tongues corresponding to the locking of different pairs of UPOs lie in a relatively narrow region. The approach to CPS thus is associated to an effective saddle-node bifurcation with small noise and a type-I intermittency occurs, resulting in the relatively hard transition to CPS. For the funnel chaotic attractors the distribution of the mean periods of UPOs is rather broad [see Figs. 1(h) and 1(i)]. Therefore the coupling strengths corresponding to the onset of

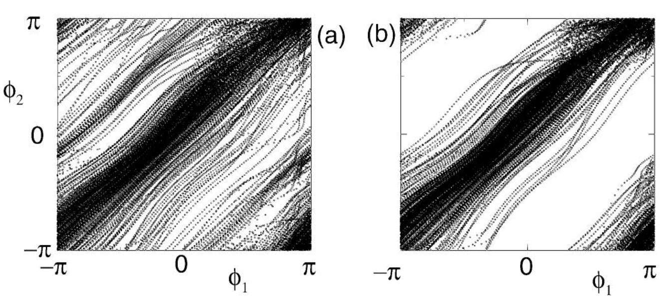

FIG. 4. Projections of trajectories of the Rössler systems for $a = 0.22$ on the plane (ϕ_1, ϕ_2) for the coupling strength d outside (a) ($d = 0.055$) and within (b) ($d = 0.075$) the synchronization region.

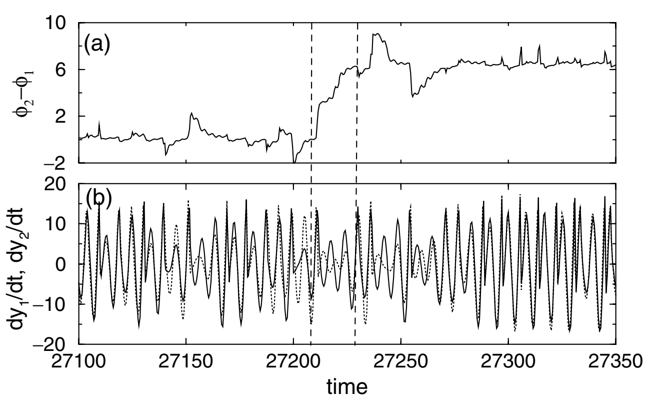

FIG. 5. (a) Time evolution of phase difference. (b) Variables $\dot{y}_{1,2}$ in system (1) for $a = 0.2925$ and $d = 0.179 \in [d_{l_3}, d_{l_1}]$. Solid and dotted lines correspond to the first and second oscillators, respectively. In the time interval between dashed lines the first oscillator produces four rotations in the $(\dot{x}_1, \dot{y}_1)$ plane around the origin, but the second oscillator generates only three rotations, which leads to a phase slip in (a).

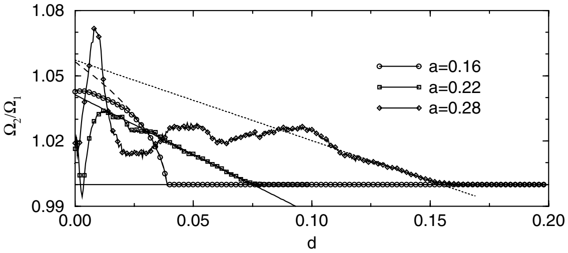

FIG. 6. Mean frequency ratio Ω_2/Ω_1 versus coupling. The fitting curve ($\sim |d - d_{l_1}|^{0.5}$) for $a = 0.16$ (dashed line). The fitting straight lines for $a = 0.22$ (solid line) and $a = 0.28$ (dotted line) are presented.

synchronization of different pairs of UPOs are distributed in a rather large interval. With the increase of d, the system crosses continuously a series of well-separated Arnold tongues, resulting in a much slower convergence to CPS.

In conclusion, we have proposed a general approach to measure the phase of any dynamical systems based on the curvature of the projection of trajectories on some plane, provided that the curvature is positive. Therefore, synchronization in phase can be observed in those chaotic oscillators where such a plane exists. Using this approach, we have observed three types of transition to CPS in dependence on the coherence properties of motions measured by diffusion of the phase. For small diffusion the onset of CPS is accompanied by the transition of one of the zero Lyapunov exponents to negative values. If the diffusion of the phase is rather strong, phase locking occurs only after the onset of GCS. For intermediate diffusion CPS sets in via an interior crises of the hyperchaotic attractor. The numerical results indicate that only these three routes to CPS are possible in systems of coupled chaotic Rössler oscillators. Our numerical experiments (not presented here) with other chaotic dynamical systems [7–12] (see also [18]) and recent experimental study with coupled chemical oscillators [19] show similar routes to CPS. We believe that these three types of transition to CPS can be also observed in many other coupled oscillators depending on the degree of phase coherence, and are of practical importance in experimental study of CPS, especially for the purpose of experimental controlling and the estimation of necessary coupling strength. The study of noise influence on the hierarchy of transitions to CPS as well as the transitions to m:n synchronization and an extension to large ensembles of coupled oscillators is a subject of future study.

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