Black Holes in Gödel Universes and pp Waves

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We find exact solutions for rotating and nonrotating neutral black holes in the Gödel universe of five-dimensional minimal supergravity theory. We also describe the embedding of this solution in M-theory. After dimensional reduction and T-duality, we obtain a supergravity solution corresponding to placing a black string in a pp-wave background.

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The Gödel universe [1] is an exact solution of Einstein's equations in the presence of a cosmological constant and homogeneous pressureless matter. This space-time solution exhibits several peculiar features including the presence of closed timelike curve and the absence of globally spatial-like Cauchy surface. Recently, a space-time exhibiting most of the peculiar features of Gödel universes was shown to be an exact solution of minimal supergravity in $4 + 1$ dimensions, preserving some number of supersymmetries [2,3]. As a result, these solutions can be embedded in supergravity theories of 10 or 11 dimensions and may constitute consistent backgrounds of string theory. The consistency of this solution was further investigated recently by [4,5]; they found that the supersymmetric Gödel universes of [2,3] are related by T-duality to the pp-wave solutions [6] which have generated significant interest recently in light of the fact that the world sheet formulation of string theory is highly tractable [7] and the fact that this string theory admits a dual field theory description [8].

In this Letter we construct a space-time describing a Schwarzschild black hole localized inside the Gödel universe, and describe some of its basic properties. Just as for Minkowski, anti de Sitter, and de Sitter spaces, such a solution provides important insights into the nature of Gödel universes. We should mention that a different but very interesting space-time describing a charged extremal black hole with finite horizon area in a Gödel universe was identified in [3].

Finding space-times describing a black object localized inside the pp wave is an important outstanding problem. Discussions of the recent approaches to this problem can be found in [9,10,11]. As a bonus for finding the black-hole solution in Gödel universes, we are able to construct the black string solution in pp waves.

We will follow [2,3] and work mostly with the bosonic components of the $4 + 1$-dimensional minimal supergravity. Unlike [2,3], however, we will not require our solutions to preserve any supersymmetry. The bosonic fields of the minimal supergravity theory in $4 + 1$ dimensions consist of the metric and a 1-form gauge field. Their equations of motion are given by

$$R_{\mu\nu} = 2\Big(F_{\mu\alpha}F_{\nu}^{\alpha} - \frac{1}{6}g_{\mu\nu}F^2\Big),$$
$$D_{\mu}F^{\mu\nu} = \frac{1}{2\sqrt{3}}\tilde{\epsilon}^{\alpha\beta\gamma\mu\nu}F_{\alpha\beta}F_{\gamma\mu}, \tag{1}$$

where $\tilde{\epsilon}_{\mu\nu\lambda\kappa\sigma} = \sqrt{-\det g}\,\epsilon_{\mu\nu\lambda\kappa\sigma}$. We follow mostly the conventions of [2,3] and use angular coordinates

$$x^1 + ix^2 = r\cos\frac{\theta}{2}e^{i[(\psi+\phi)/2]},$$
$$x^3 + ix^4 = r\sin\frac{\theta}{2}e^{i[(\psi-\phi)/2]}. \tag{2}$$

The Gödel universe is a solution of the equations of motion (1) given by

$$ds^2 = -(dt + jr^2\sigma_L^3)^2 + dr^2 + \frac{r^2}{4}(d\theta^2 + d\psi^2 + d\phi^2 + 2\cos\theta\, d\psi d\phi), \tag{3}$$
$$A = \frac{\sqrt{3}}{2}jr^2\sigma_L^3,$$

where, following the conventions of [2,3],

$$\sigma_L^3 = d\phi + \cos\theta\, d\psi. \tag{4}$$

The parameter j sets the scale of the background, which reduces to Minkowski space for small j. From the sign of the $g_{\phi\phi}$ component of the metric, it is easy to recognize a closed timelike curve parametrized by ϕ with all other coordinates fixed, for $r > 1/2j$. We will refer to the surface at fixed r where $g_{\phi\phi}$ vanishes as the "velocity of light surface." It should be emphasized, however, that since the Gödel space-time is homogeneous, there is a closed timelike curve going through every point in space-time.

In order to find a solution to the supergravity equations of motion describing black holes in this background, consider an ansatz

$$ds^2 = -f(r)\,dt^2 - g(r)\,r\sigma_L^3 dt - h(r)\,r^2(\sigma_L^3)^2 + k(r)\,dr^2 + \frac{r^2}{4}(d\theta^2 + d\psi^2 + d\phi^2 + 2\cos\theta\,d\psi d\phi),$$
$$A = \frac{\sqrt{3}}{2} j r^2 \sigma_L^3. \tag{5}$$

Substituting this ansatz into the equations of motion gives rise to a rather complicated set of equations. It can be easily checked, however, that the choice

$$f(r) = 1 - \frac{2m}{r^2}, \qquad g(r) = 2jr,$$
$$h(r) = j^2(r^2 + 2m), \qquad k(r) = \left(1 - \frac{2m}{r^2} + \frac{16j^2m^2}{r^2}\right)^{-1} \tag{6}$$

solves these equations. In the small j limit, this solution reduces to an ordinary Schwarzschild black hole in $4+1$ dimensions. On the other hand, in the small m limit, we recover the Gödel universe. We therefore conclude that this background corresponds to a Schwarzschild black hole placed inside the Gödel universe. This solution is the main result of this Letter; we will study its various properties.

The ansatz (5) clearly preserves five of the nine isometries of the Gödel universe. They are generated by time translation ∂_t, as well as the $SU(2) \times U(1)$ subgroup of the $SO(4) = SU(2) \times SU(2)$ isometry group on S^3 ,

$$\xi_1^R = -\cot\theta\cos\psi\,\partial_\psi - \sin\psi\,\partial_\theta + \frac{\cos\psi}{\sin\theta}\,\partial_\phi,$$
$$\xi_2^R = -\cot\theta\sin\psi\,\partial_\psi + \cos\psi\,\partial_\theta + \frac{\sin\psi}{\sin\theta}\,\partial_\phi, \tag{7}$$
$$\xi_3^R = \partial_\psi, \qquad \xi_3^L = \partial_\phi.$$

The most salient feature of the solution (6) is the existence of a horizon at

$$r_{\rm BH}^2 = 2m(1 - 8j^2m). \tag{8}$$

This surface is a horizon in the sense that future directed light cones emanating from all points inside the horizon are strictly contained inside. The surface $r = r_{\rm BH}$ also exhibits many of the standard properties of a black-hole horizon, as we will see below.

The area of the horizon is

$$\mathcal{A} = 2\pi^2\sqrt{8m^3(1 - 8j^2m)^5}. \tag{9}$$

It is tempting to apply the standard interpretation of black-hole thermodynamics and think of

$$S = \frac{\mathcal{A}}{4G_5} \tag{10}$$

as the entropy. It is interesting to note that with this definition the entropy does not increase monotonically with m. In fact, at $m = 1/8j^2$, the horizon area vanishes. For m greater than this critical value, the horizon disappears and the space-time (6) is nakedly singular. This appears to constitute a physical upper bound on a mass of a neutral state in a Gödel universe, rendering the number of degrees of freedom to be finite.

As one increases m, the velocity of light surface also moves in toward small r,

$$r_v^2 = \frac{1}{4j^2}(1 - 8j^2m). \tag{11}$$

One of the main objectives of [4] was to study the holography of Gödel universes from a "phenomenological" point of view and to compare against the known properties of de Sitter space. To this end, the authors of [4] computed the location of a "preferred holographic screen" for a timelike observer at the origin, as defined in [12]. The area of the preferred screen is a measure of the number of degrees of freedom contained in the volume enclosed by this surface. Because of the SU(2) symmetry (7), this simply amounts to computing the maximum area for a family of surfaces with 3-sphere topology, each at fixed t and r coordinates. For the space-time (6), each surface has area

$$\mathcal{A}\,(r) = r^3\sqrt{1 - 4j^2(r^2 + 2m)}. \tag{12}$$

For $m = 0$, this expression reduces to the result obtained in [4]. The location and area of the preferred holographic screen come out to be

$$r_s^2 = \frac{3(1 - 8j^2m)}{16j^2}, \qquad \mathcal{A}_s = \frac{3\sqrt{3}\pi^2(1 - 8j^2m)^2}{64j^3}. \tag{13}$$

Note that the area $\mathcal{A}_s$ of the preferred screen decreases as we increase m. Also, at a finite value of $m = 3/32j^2$, but before the critical value for the appearance of a naked singularity $m = 1/8j^2$, the black-hole horizon and the preferred screen coincide. This convergence is similar to what was observed in de Sitter space [13], except that in that case screen and horizon meet just as a naked singularity appears.

The curvature invariants for this space-time take a simple form. For example, the Ricci scalar is

$$R = \frac{16j^2(r^2 - m)}{r^2}. \tag{14}$$

The space-time is smooth at the horizon radius as long as $m < 1/8j^2$, but there is always a singularity at $r = 0$ for nonvanishing m.

It possible to find an analytic form for geodesics in this space-time geometry following the approach of [4]. Let us write the tangent vector to the geodesic as

$$\xi = \dot{t}\frac{\partial}{\partial t} + \dot{r}\frac{\partial}{\partial r} + \dot{\theta}\frac{\partial}{\partial \theta} + \dot{\psi}\frac{\partial}{\partial \psi} + \dot{\phi}\frac{\partial}{\partial \phi}, \tag{15}$$

where the dot denotes derivative $d/d\lambda$ with respect to the affine parameter λ, and define the integrals of motion

$$(\xi, \xi) = -M^2, \qquad (\xi, \partial_t) = -E,$$
$$(\xi, \partial_\psi) = L_\psi, \qquad (\xi, \partial_\phi) = L_\phi. \tag{16}$$

It is useful to define one more integral of motion

$$\Xi = (\xi, \xi_1^R) = \cos\psi\left(L_\psi \cos\theta - \frac{L_\phi}{\sin\theta}\right) - \frac{r^2\dot{\theta}^2 \sin(\phi)}{4}, \tag{17}$$

where ξ_1^R is one of the isometries listed in (7). The constraints implied by these conserved quantities simplify drastically if we set $L_\psi = L_\phi = \Xi = 0$ and read

$$\dot{t} = \frac{r^2(1 - 4j^2(r^2 + 2m))}{r^2 - 2m + 16j^2m^2}, \qquad \dot{\theta} = 0,$$
$$\dot{\psi} = 0, \qquad \dot{\phi} = \frac{4jr^2}{r^2 - 2m + 16j^2m^2}. \tag{18}$$

If we set $M = 0$ so as to consider only null geodesics, we find a very simple expression for the radial derivative

$$\dot{r}^2 = (1 - 8j^2m) - 4j^2r^2, \tag{19}$$

which is easily solved. Integrating (18) then gives the full expression for the geodesics in this family. If we pick initial conditions so that

$$r(\lambda) = \frac{1}{2j}\sqrt{1 - 8j^2m}\cos(2j\lambda), \tag{20}$$

the corresponding geodesic is tangent to the velocity of light surface at $\lambda = 0$ and asymptotes to a trajectory

$$r^2 = r_{\rm BH}^2 = 2m(1 - 8j^2m),$$
$$\phi = \frac{4j}{(1 - 8j^2m)^2}t + \phi_0, \tag{21}$$

which spirals into the horizon in infinite coordinate time t (although it crosses the horizon in finite affine time); see Fig. 1 for an illustration. This is the behavior expected for a geodesic falling toward a black-hole horizon. One can also compute the expansion scalar for the congruence associated with this family of geodesics,

$$\Theta = \frac{3 - 8j^2(2r^2 + 3m)}{r\sqrt{1 - 4j^2(r^2 + 2m)}}, \tag{22}$$

which vanishes at the radius of the preferred screen $r = r_s$ as expected.

It is interesting to consider the embedding of the solution (6) of 4 + 1-dimensional supergravity into string theory. The most efficient way to do this is to first embed the solution of supergravity in 4 + 1 dimensions into 10 + 1 dimensions. This can be done by considering an ansatz

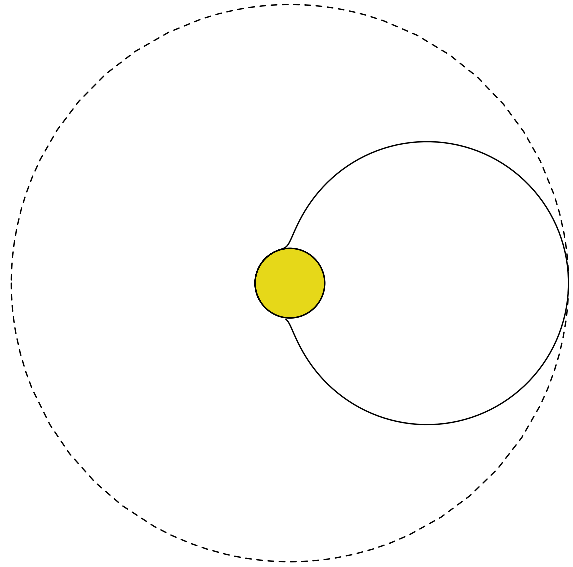

FIG. 1 (color online). The projection to fixed (t, θ, ψ) plane of a null geodesic which is tangent to the velocity of light surface at an instant and spirals in toward the horizon in the future and in the past. The dotted line is the velocity of light surface and the shaded region is the region inside the horizon.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \sum_{i=5}^{10}(dx^i)^2, \qquad C = \frac{2}{\sqrt{3}}A \wedge K, \tag{23}$$

where

$$K = dx^5 \wedge dx^6 + dx^7 \wedge dx^8 + dx^9 \wedge dx^{10}. \tag{24}$$

Indices μ and ν run from 0 to 4. One can verify that substituting this ansatz into the supergravity equations of motion in 10 + 1 dimensions,

$$R_{ab} - \frac{1}{12}\left(F_{acde}F_b{}^{cde} - \frac{1}{12}g_{ab}F^2\right), \tag{25}$$

gives rise precisely to the equations of motion for supergravity in 4 + 1 dimensions (1). One can now dimensionally reduce along the x^{10} coordinate to obtain the solution (6) embedded into type IIA supergravity.

In the absence of the black hole, T-dualizing along x^9 will give rise to a pp-wave solution obtained by taking the Penrose limit of intersecting D3-branes. In the presence of the black hole, T-duality yields a new black string solution for the type IIB theory,

$$ds^2 = -\left(1 - \frac{2m}{r^2}\right)dt^2 + dy^2 - 2jr^2\sigma_L^3(dt - dy) - 2mj^2r^2(\sigma_L^3)^2 + \left(1 - \frac{2m}{r^2} + \frac{16j^2m^2}{r^2}\right)^{-1}dr^2$$
$$+ \frac{r^2}{4}(d\theta^2 + d\psi^2 + d\phi^2 + 2\cos\theta\, d\psi d\phi) + ds_{T^4}^2, \tag{26}$$

where y is the coordinate dual to x^9. There are also nontrivial field strengths for the antisymmetric tensor fields in this background which can be found by following the duality starting from (23).

In the limit $m \to 0$, this new solution (26) reduces to a well known pp-wave geometry. In the $j \to 0$ limit, the solution reduces to uncharged black string solution of type IIB supergravity. This suggests that (26) should be interpreted as the solution describing a black string in an asymptotically pp-wave background geometry.

There is a subtlety with this interpretation, however. If one collects the angular part of the metric (26) by fixing t, y, and r, one finds

$$ds^2 = \frac{r^2}{4}[(\sigma_L^1)^2 + (\sigma_L^2)^2 + (1 - 8j^2 m)(\sigma_L^3)^2]. \quad (27)$$

In other words, the term in the deformed metric along σ_L^3 is of the same order in r as the undeformed metric of the round 3-sphere. Turning on m therefore has the effect of squashing this 3-sphere until one reaches $m = 1/8j^2$ at which point the 3-sphere is squashed completely. Because of this effect on the asymptotic geometry, the solution (26) cannot be interpreted as that of a black string with the same asymptotic background geometry as the empty pp wave.

An important question is whether or not it is possible to find a solution that would describe a black string which does not squash or otherwise affect the large r asymptotic geometry of the pp wave. It may be that a black hole, as opposed to a black string, will have smaller effect on this asymptotic geometry. Similar gravitational back reaction effects due to localized and delocalized extremal sources were observed in [14]. Perhaps finite energy density uniformly distributed along the light-cone coordinates of the pp wave generically back reacts to deform the large r asymptotics. It would also be interesting to examine how the Laflamme-Gregory instability of the black string solution [15] is affected when j is nonvanishing.

It would also be interesting to explore various generalizations of the solution (6). For example, one can easily verify that

$$f(r) = 1 - \frac{2m}{r^2}, \qquad g(r) = 2jr + \frac{2ml}{r^3},$$

$$h(r) = j^2(r^2 + 2m) - \frac{ml^2}{2r^4}, \quad (28)$$

$$k(r) = \left(1 - \frac{2m}{r^2} + \frac{16j^2m^2}{r^2} + \frac{8jml}{r^2} + \frac{2ml^2}{r^4}\right)^{-1}$$

also solves the equations of motion. By taking the $j \to 0$ limit, we recover the rotating black-hole solution of $4 + 1$ gravity [16]. One could therefore think of this solution as a rotating black hole inside the Gödel universe. It is possible, in particular, to tune the angular momentum

$$l = -4jm \quad (29)$$

so as to make the horizon angular velocity Ω_H vanish.

It should also be possible to find a generalization to charged black holes by following the construction outlined in [17,18,19]. This will give rise to the nonextremal generalization of the black-hole solution identified in [3].

There are many other interesting issues to explore. One would like to map out the full causal structure of the black-hole solution (6). It would also be interesting to find a suitable generalization of Arnowitt-Deser-Misner (ADM) mass and angular momentum to the Gödel universe and to compute their values for (6) and (28). Ultimately, one would like to understand the physical meaning of the area of the preferred screen (13) and the horizon (9) as being related to some microscopic state counting, possibly of strings in a pp-wave background. We hope that the explicit solution of supergravity equations of motion describing black holes in this background will stimulate further insight into these fascinating issues.

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