

## Front Dynamics in Reaction-Diffusion Systems with Levy Flights: A Fractional Diffusion Approach

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(Received 17 December 2002; published 3 July 2003)

The use of reaction-diffusion models rests on the key assumption that the diffusive process is Gaussian. However, a growing number of studies have pointed out the presence of anomalous diffusion, and there is a need to understand reactive systems in the presence of this type of non-Gaussian diffusion. Here we study front dynamics in reaction-diffusion systems where anomalous diffusion is due to asymmetric Levy flights. Our approach consists of replacing the Laplacian diffusion operator by a fractional diffusion operator of order  $\alpha$ , whose fundamental solutions are Levy  $\alpha$ -stable distributions that exhibit power law decay,  $x^{-(1+\alpha)}$ . Numerical simulations of the fractional Fisher-Kolmogorov equation and analytical arguments show that anomalous diffusion leads to the exponential acceleration of the front and a universal power law decay,  $x^{-\alpha}$ , of the front's tail.

DOI: 10.1103/PhysRevLett.91.018302

PACS numbers: 82.40.Ck, 02.50.Ey, 05.40.Fb

Reaction-diffusion models have found widespread applicability in many areas, including chemistry, biology, physics, and engineering; see, for example, Ref. [1] and references therein. The simplest reaction-diffusion models are of the form

$$\partial_t \phi = \chi \partial_x^2 \phi + F(\phi), \quad (1)$$

where  $\chi$  is the diffusion constant, and  $F$  is a nonlinear function representing the reaction kinetics. Examples of particular interest include the Fisher-Kolmogorov equation for which  $F = \gamma\phi(1 - \phi)$  and the real Ginzburg-Landau equation for which  $F = \gamma\phi(1 - \phi^2)$ . The nontrivial dynamics of these systems arises from the competition between the reaction kinetics and diffusion.

At a microscopic level, diffusion is the result of the random motion of individual particles, and the use of Laplacian operators  $\partial_x^2 \phi$  to model it rests on the key assumption that this random motion is a stochastic Gaussian process. However, a growing number of works have shown the presence of anomalous diffusion processes for which the mean square variance  $\langle [x - \langle x \rangle]^2 \rangle \sim t^\gamma$  grows faster ( $\gamma > 1$ , in the case of superdiffusion) or slower ( $\gamma < 1$ , in the case of subdiffusion) than in a Gaussian diffusion process [2]. Accordingly, an important open problem is to understand the dynamics of reaction-diffusion systems when the assumption of Gaussian diffusion fails. This problem has a particular relevance to plasma physics, which was our original motivation to carry out this study. In particular, reaction-diffusion models with Gaussian diffusion have been used to study the dynamics of spatiotemporal propagating fronts in the transition to high confinement regimes in magnetically confined plasmas [3]. However, these studies must incorporate the fact that, in this type of plasma, evidence of non-Gaussian diffusion has been observed in perturbative transport experiments [4], in numerical

simulation of three-dimensional turbulence [5], and in test-particle transport studies [6].

The origin of non-Gaussian diffusion can be traced back to the existence of long-range correlations in the dynamics, or the presence of anomalously large particle displacements described by broad probability distributions. Here we are interested in the second possibility. In particular, we focus on systems that exhibit anomalous diffusion caused by Levy flights, for which the probability distribution of particle displacements,  $p(\ell)$ , is broad in the sense that  $\langle \ell^2 \rangle = \infty$ . As is well known, for these kinds of systems the central limit theorem cannot be applied; and as  $N \rightarrow \infty$ , the probability distribution function of  $x = \sum_n^N \ell_n$ , rather than being Gaussian, is an  $\alpha$ -stable Levy distribution [2,7].

The majority of studies on anomalous diffusion due to Levy flights have focused on symmetric processes for which  $p(\ell) = p(-\ell)$ . However, this is not always the case. For example, numerical studies of test-particle transport in magnetized plasmas and in geophysical flows show that these systems have a built-in asymmetry that gives rise to asymmetric transport [6,8]. Also, non-Gaussian asymmetric processes are likely to have an important role in the asymmetries observed in pulse propagation experiments in confined plasmas [4]. Motivated by this, we focus on anomalous diffusion processes that exhibit Levy flights in one direction, say, for  $x > 0$ , but Gaussian behavior in the other direction,  $x < 0$ . The main goal of this Letter is to understand how this asymmetry manifests in the propagation of right-moving ( $x > 0$ ) and left-moving ( $x < 0$ ) fronts in reaction-diffusion systems. To study this problem we propose the following model:

$$\partial_t \phi = \chi_a D_x^\alpha \phi + F(\phi), \quad (2)$$

$${}_a D_x^\alpha \phi = \frac{1}{\Gamma(2 - \alpha)} \partial_x^2 \int_a^x \frac{\phi(y)}{(x - y)^{\alpha-1}} dy, \quad (3)$$

where the Laplacian operator  $\partial_x^2$  has been replaced by  ${}_a D_x^\alpha$ , the Riemann-Liouville, left-fractional derivative of order  $\alpha$ , with  $1 \leq \alpha < 2$ . Fractional calculus is a natural mathematical generalization of standard calculus [9], for example,  ${}_{-\infty} D_x^\alpha e^{ikx} = (ik)^\alpha e^{ikx}$ , and in recent years has been applied to a large class of problems in the applied sciences; see Refs. [10,11], and references therein.

Previous studies of reaction-diffusion systems with anomalous diffusion include Ref. [12], where reaction in the presence of subdiffusion was studied using fractional derivative operators in time. In contrast, here we are interested in superdiffusion, which we model with fractional operators in space. The interplay of bistable reaction processes and anomalous diffusion caused by Levy flights was addressed in Ref. [13]. More recently, superfast front propagation in reactive systems with non-Gaussian diffusion was discussed in Ref. [14]. The model studied in Ref. [14] consisted of a time-discrete reaction system coupled to a superdiffusive Levy process described by an integral operator with an algebraic decaying propagator. In our model in Eq. (2), we do not assume a time-discrete reaction kinetics; and through the use of fractional operators we use the exact Levy propagator. In addition, we consider here the role of asymmetric transport.

The physical motivation for using the left-fractional derivative rests on the fact that for  $F = 0$ , Eq. (2) reduces to an asymmetric, space-fractional diffusion equation whose solution for a delta function initial condition,  $\phi(x, t = 0) = \delta(x)$ , in the infinity domain  $x \in (-\infty, \infty)$  is

$$\phi(x, t) = \frac{1}{(\chi t)^{1/\alpha}} P_\alpha \left[ \frac{x}{(\chi t)^{1/\alpha}} \right], \quad (4)$$

where

$$P_\alpha(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i^\alpha k^\alpha + ik\eta} dk. \quad (5)$$

For  $\alpha = 2$ , Eq. (5) is the Gaussian propagator, and Eq. (4) is the fundamental solution of the standard diffusion equation. However, for  $1 < \alpha < 2$ , Eq. (5) is an extremal,  $\alpha$ -stable Levy distribution, i.e., a distribution with maximum skewness. These distributions are the attractors of stochastic processes that exhibit Levy flights in only one direction, they have algebraic asymptotic behavior at plus infinity,  $p_\alpha(\eta) \sim 1/\eta^{\alpha+1}$ , but decay exponentially at minus infinity [7]. In principle, one could add to Eq. (2) a right-fractional derivative with a suitable weighting factor, and for  $F = 0$  obtain Levy  $\alpha$ -stable distribution of arbitrary skewness [15]. Equation (4) implies that  $\langle x^n \rangle = (\chi t)^{n/\alpha} \int_{-r}^r \eta^n p_\alpha(\eta) d\eta$  diverges as  $r \rightarrow \infty$ , for  $n \geq \alpha$ . However, in physical applications (e.g., Ref. [8]), a finite- $r$  cutoff leads to the finite-size scaling  $\langle x^n \rangle \sim t^{n/\alpha}$ , which implies superdiffusive behavior with  $\gamma = 2/\alpha$ .

In the remainder of this Letter, we present a numerical and analytical study of front solutions of Eq. (2) with a reaction dynamics of the Fisher-Kolmogorov type,  $F =$

$\gamma\phi(1 - \phi)$ , with two fixed points, one stable,  $\phi = 1$ , and the other unstable,  $\phi = 0$ . As it is usually done for initial value problems with fractional derivatives in time, and boundary value problems in finite domains with fractional derivatives in space [9,10], we regularize the singular behavior of the fractional operator by subtracting the value of  $\phi(x)$  at the lower limit, namely

$$\partial_t \phi = \chi_a D_x^\alpha [\phi - \phi(a)] + \gamma\phi(1 - \phi), \quad (6)$$

where  $\phi(a) = \phi(x = a, t)$ . In an infinite domain,  $a \rightarrow -\infty$ , and Eq. (6) reduces to Eq. (2) since the fractional derivative,  ${}_{-\infty} D_x^\alpha$ , of a constant is zero. Here we consider the finite domain,  $x \in (0, 1)$ , and take  $a = 0$ . The numerical solutions were obtained by integrating Eq. (6) with boundary conditions  $\phi(0) = 1$  and  $\phi'(0) = 0$  for right-propagating fronts, and  $\phi'(0) = 0$  and  $\phi(1) = 1$  for left-propagating fronts. We used a semi-implicit time advance with an up-wind finite difference scheme. The fractional operator was discretized using the Grunwald-Letnikov definition of the fractional derivative [9]. Details on the numerical method will be published elsewhere.

We consider two classes of initial conditions

$$\phi_0^{r,l}(x) = \frac{1}{2} \left[ 1 \mp \tanh \left( \frac{x - x_0}{W} \right) \right], \quad (7)$$

where  $\phi_0^r(x)$  takes the  $-$  sign and corresponds to a right-propagating front, and  $\phi_0^l(x)$  takes the  $+$  sign and corresponds to a left-propagating front. Figure 1 shows the time evolution of the front profile for  $\phi_0^r(x)$  with  $W = 0.001$  and  $x_0 = 0.003$ , obtained from the direct numerical integration of Eq. (6) with  $\gamma = 1$  and  $\chi = 5 \times 10^{-7}$ . In this case, the front propagates to the right and develops an algebraic decaying tail  $\phi \sim x^{-\alpha}$ . The time evolution of the tail,  $\phi(x = 1, t)$ , exhibits, for large  $t$ , exponential

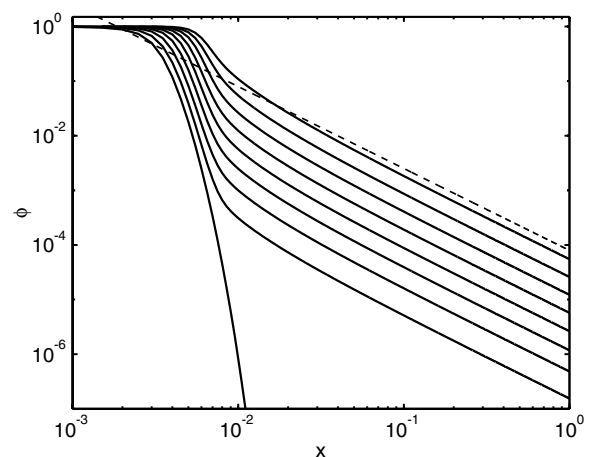


FIG. 1. Right-propagating front profiles at successive times, obtained from a numerical integration of the fractional Fisher-Kolmogorov Eq. (6) with  $\alpha = 1.5$  and initial condition  $\phi(x, 0) = \phi_0^r(x)$  in Eq. (7). The dashed line has a slope equal to  $\alpha$ . In agreement with the analytical result, right-moving fronts develop an algebraically decaying tail,  $\phi \sim x^{-\alpha}$ .

growth  $\phi \sim e^{\gamma t}$ . The acceleration of the front is evident in the space-time diagram in Fig. 2 that shows a contour plot of  $\phi(x, t)$ . Front acceleration and algebraic decay of the tail was also observed in the model studied in Ref. [14], but with a different exponent, namely  $\phi \sim x^{-(\alpha+1)}$ .

Left-propagating fronts exhibit a more standard dynamics. In particular, Fig. 3 shows the time evolution of the front profile for an initial condition  $\phi_0^l(x)$  with  $W = 0.001$ ,  $\gamma = 1$ , and  $\chi = 5 \times 10^{-7}$  (the same parameter values used in Fig. 1) and  $x_0 = 0.9$ . In this case, the front exhibits an exponential decay and a self-similar propagation with constant speed  $c$ .

The numerical results presented above can be explained analytically using the leading-edge approximation, extensively used in the study of fronts with Gaussian diffusion [16]. In this approximation, in the limit  $\phi \ll 1$ , the reaction kinetics  $F(\phi)$  is linearized around the unstable phase leading to the linear fractional equation:

$$\partial_t \phi = \chi_{-\infty} D_x^\alpha \phi + \gamma \phi. \quad (8)$$

Substituting  $\phi = e^{\gamma t} \psi(x, t)$  into Eq. (8) yields a fractional-diffusion equation for  $\psi$  whose general solution is

$$\psi(x, t) = \int_{-\infty}^{\infty} p_\alpha(\eta) \psi_0[x - (\chi t)^{1/\alpha} \eta] d\eta, \quad (9)$$

where, as discussed before,  $p_\alpha(\eta)$  given in Eq. (5) is the extremal Levy  $\alpha$ -stable distribution that is the Green's function of the asymmetric fractional-diffusion equation. For a front initial condition of the form  $\phi_0(x < 0) = 1$  and  $\phi_0(x > 0) = e^{-\lambda x}$ , Eq. (9) gives

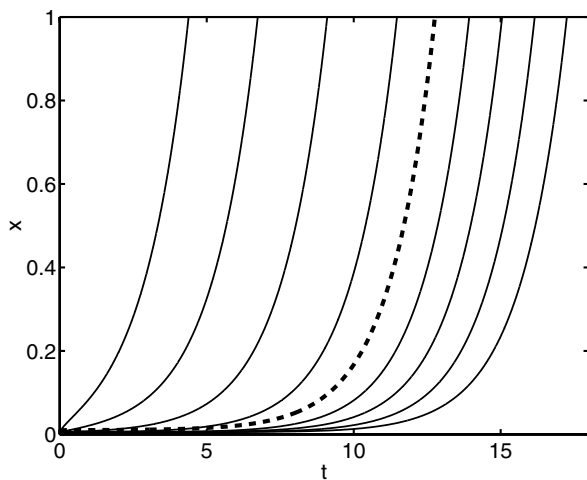


FIG. 2. Contour plot of the numerical solution  $\phi(x, t)$  in Fig. 1. The curvature of the isocontours illustrates the exponential acceleration of fronts in the fractional Fisher-Kolmogorov equation. The dashed line corresponds to the analytical scaling result  $\phi \sim x^{-\alpha} e^{\gamma t}$ .

$$\begin{aligned} \phi(x, t) = & e^{\gamma t} \int_{x(\chi t)^{-1/\alpha}}^{\infty} p_\alpha(\eta) d\eta \\ & + e^{-\lambda x + \gamma t} \int_{-\infty}^{x(\chi t)^{-1/\alpha}} e^{\lambda(\chi t)^{1/\alpha} \eta} p_\alpha(\eta) d\eta. \end{aligned} \quad (10)$$

Before using this solution to obtain the asymptotic behavior of fractional-diffusion, right-propagating fronts, it is instructive to consider the standard diffusion limit,  $\alpha = 2$ . In this case, the Green's function is the Gaussian propagator  $p_2(\eta)$ , and Eq. (10) becomes

$$\phi(x, t) = e^{-\lambda(x-ct)} P\left(\frac{x-2\lambda\chi t}{\sqrt{2\chi t}}\right) + e^{\gamma t} \left[1 - P\left(\frac{x}{\sqrt{2\chi t}}\right)\right], \quad (11)$$

where  $P(z)$  is the normal probability distribution function. Using the fact that  $P(z \rightarrow \infty) = 1$ , we get from Eq. (11) the asymptotic behavior  $\phi(x, t) \sim e^{-\lambda(x-ct)}$ . Where the speed of the front,  $c$ , is related to the steepness of the front,  $\lambda$ , according to  $c = \gamma/\lambda + \chi\lambda$ , with the minimum front speed  $c_m = 2\sqrt{\gamma\chi}$  selected for sufficiently steep profiles [1,16].

In the fractional case we consider a large, fixed  $t$ , and  $x(\chi t)^{-1/\alpha} \rightarrow \infty$ . Introducing a cutoff  $\Omega$  such that  $1 \ll \Omega < x(\chi t)^{-1/\alpha}$  in the second integral in Eq. (10), integrating by parts, and using the asymptotic expression of the Levy distribution,  $p_\alpha(\eta) \sim 1/\eta^{\alpha+1}$  we get

$$\begin{aligned} \phi \sim & \chi t e^{\gamma t} \left[ \frac{x^{-\alpha}}{\alpha} + \frac{x^{-\alpha-1}}{\lambda} + \frac{(1+\alpha)}{\lambda(\chi t)^{1+1/\alpha}} \right. \\ & \left. \times \int_{\Omega}^{x(\chi t)^{-1/\alpha}} \frac{e^{\lambda[(\chi t)^{1/\alpha} \eta - x]}}{\eta^{2+\alpha}} d\eta + \dots \right], \end{aligned} \quad (12)$$

where the dots denote terms of order  $e^{-\lambda x}$ . Since the

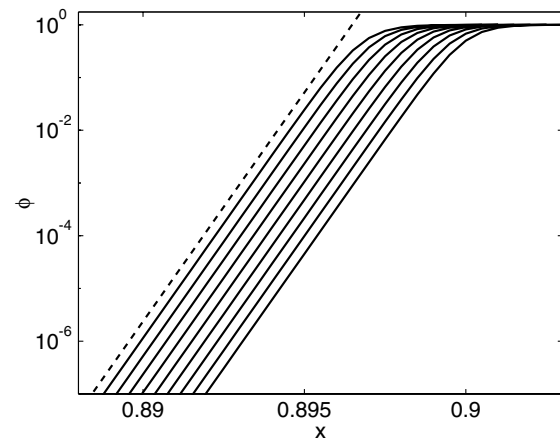


FIG. 3. Left-propagating front profiles at successive times obtained from a numerical integration of the fractional Fisher-Kolmogorov Eq. (6) with  $\alpha = 1.5$  and initial condition  $\phi(x, 0) = \phi^l(x)$  in Eq. (7). The dashed line has a slope equal to  $\lambda = 2/W = 2000$ . In agreement with the analytical result in Eq. (13), the front exhibits an exponential decay and has a speed of  $c = 5.22 \times 10^{-4}$ .

integrand on the right-hand side of Eq. (12) is bounded by  $1/\eta^{2+\alpha}$ , the third term in this equation is at most of order  $x^{-\alpha-1}$ . Therefore, in agreement with the numerical result in Fig. 1, to leading order the tail of right-propagating fronts decays as  $\phi \sim 1/x^\alpha$ . The time-asymptotic dynamics for fixed, large  $x$  can also be obtained from Eq. (10). In this case, a stationary phase approximation gives, in agreement with the numerical results,  $\phi \sim e^{\gamma t}$  to leading order, with a correction of order  $(\chi t)^{-1/\alpha} e^{\gamma t}$ .

The previously discussed asymptotic results can be summarized as  $\phi \sim x^{-\alpha} e^{\gamma t}$ , which, as Fig. 2 shows, reproduces the numerical results. Solving the equation  $\phi(x, t) = \phi_L$  for  $x$ , for a fixed value  $\phi = \phi_L \in (0, 1)$  gives the Lagrangian trajectory  $x_L = x(t; \phi_L)$  of the coordinate of a point in the front with concentration  $\phi = \phi_L$ . In the asymptotic limit we have  $x_L \sim e^{\gamma t/\alpha}$ . That is, the Lagrangian velocity  $v_L = dx_L/dt$  of right-moving fronts grows exponentially with time,  $v_L \sim e^{\gamma t/\alpha}$ .

To conclude, consider the dynamics of left-moving fronts, such as the one shown in Fig. 3. In this case, the leading-edge description is straightforward since these fronts exhibit exponential tails and propagate at constant speed. Substituting  $\phi \sim \exp[\lambda(x + ct)]$  into Eq. (2), we obtain the dispersion relation  $c = c(\lambda)$ , with minimal front speed  $c_m = c(\lambda_m)$  where

$$c = \frac{\gamma}{\lambda} + \frac{\chi}{\lambda^{1-\alpha}}, \quad c_m = \alpha \chi^{1/\alpha} \left( \frac{\gamma}{\alpha - 1} \right)^{(\alpha-1)/\alpha}. \quad (13)$$

Numerical results support the idea that for steep ( $\lambda \geq \lambda_m$ ) initial conditions the front selects the minimum velocity  $c_m$ , whereas for wide ( $\lambda < \lambda_m$ ) initial conditions the front speed depends on the initial condition according the first equation in (13). As expected, in the limit  $\alpha = 2$  the well-known results of front propagation in the presence of Gaussian diffusion [1,16] are recovered. Summarizing, in this Letter we have proposed the use of fractional-diffusion operators to study front dynamics in reaction-diffusion systems with non-Gaussian diffusion caused by asymmetric Levy flights. Numerical and analytical results show that right-moving fronts accelerate exponentially, and develop an algebraic decaying tail,  $\phi \sim x^{-\alpha} e^{\gamma t}$ . Left-moving fronts have exponential decaying tails and move at a constant speed given by Eq. (13). The results are general in the sense that they are independent of the details of the reaction kinetics, provided it is of the ‘‘pull’’ type with a stable and an unstable phase. Two areas of potential application of the ideas presented here are plasma physics and biology. In particular, transport studies in three-dimensional pressure-driven plasma turbulence [5] and in drift waves [6] have shown evidence of anomalous diffusion and non-Gaussian probability distributions of particle displacements which can be modeled using fractional-diffusion equations. Also, fractional-diffusion equations seem to capture important aspects of perturbative transport experiments in fusion plasmas including nonlocal diffu-

sion effects in the propagation of cold pulses (e.g., [4]). On a parallel development, reaction-diffusion models with Gaussian diffusion have been used to study the turbulence-shear flow interaction in the L-H transition in fusion plasmas (e.g., [3]), and there is a pressing need to understand the role of non-Gaussian, nonlocal diffusion in the dynamics of L-H transition fronts. The results presented here represent a first step in the study of this open problem. On the other hand, the fractional Fisher-Kolmogorov equation discussed here shares some similarities with reaction equations with integro-differential operators used in Biology to model long-range diffusion and spatial patterns in neural firing [1]. In this regard, the algebraic decaying kernel in the fractional diffusion operator might be useful for describing strongly nonlocal processes.

We thank Angelo Vulpiani and Michael Menzinger for useful conversations. This work was sponsored by the Oak Ridge National Laboratory, managed by UT-Battelle, LLC, for the U.S. Department of Energy under Contract No. DE-AC05-00OR22725.

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