The Dynamics of Multidimensional Secession: Fixed Points and Ideological Condensation

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We explore a generalized, stochastic seceder model of societal dynamics with variable size polling groups and higher-dimensional opinion vectors, revealing its essential modes of self-organized segregation. Renormalizing to a discrete, deterministic version, we pin down the upper critical size of the sampling group and analytically uncover a self-similar hierarchy of dynamically stable, multiple-branch fixed points. In $d \geq 3$, the evolving, coarsening population suffers collapse to a 2D ideological plane.

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Dynamical phenomena in which an initially homogeneous collection of weakly interacting particles can disperse, aggregate, and form clusters arises in many different physical, biological, and sociological contexts. Condensation and droplet formation [1] is, of course, a well-known example in physics, galaxy formation and clustering [2] another. In traffic patterns [3], the real-space jams that plague highway driving are, for some, a daily reminder of such intrinsic tendencies in correlated systems far from equilibrium. The formation of swarms, schools, herds [4], or even the flocking of birds [5,6], provide compelling zoological illustrations. In these cases, joining the group yields advantages over standing alone, be it by better exploration of food resources, protection from predators, or easing the flow in aerodynamic flight. Nevertheless, sometimes, as in fashion trends and similar social (or even financial) settings, standing apart from the crowd can also be a seed for the formation of new groups, splitting off the mainstream, though maybe becoming the mainstream themselves later on. In these instances, *multiple steady-state groups can be the norm.* Such matters are manifest in recent, though now classic, implementations of Arthur's variant of the *El Farol* bar problem [7], as, for example, discussed by Challet and Zhang [8], where a multitude of competing agents, armed with limited memory strategies, compete via statistical Sisyphian dynamics to be in the minority group. Interestingly, with stochasticity introduced to the decision-making process, Johnson and co-workers [9] uncovered a tendency towards *self-organized segregation* within such evolutionary minority games. Subsequently, Hod and Nakar [10] discovered a dynamical phase transition in this setting, between two group segregation and single group clustering, driven by the economic cost-benefit ratio defined in the model. More recently, Szabó and Hauert [11] studied three group dynamics amidst cooperation, defection, and abstention in a noncompulsary public goods game. The seceder model [12] was introduced, by contrast, to demonstrate that an interative mechanism favoring *individuality* cannot only create distinct groups, but also yields a rich diversity of cluster-forming dynamics. The essential tack was to give a small advantage to individuals that distinguish themselves from others. This is not unnatural, since in epidemics, for example, genetic differences can enhance long-term survival probabilities. Likewise, for players in a minority game, distinctness may an asset. In this Letter, we consider the seceder model in its broadest sense. Its description is stunningly simple: within a population of individuals, each described by a genotype, strategy, or opinion variable, choose a subset and calculate its average. From within this selection multiplet, the individual most distant from the mean is picked. Create an offspring by taking this parent's value plus a small uniform deviate. Finally, replace a random member of the population by this new offspring. This process, *pitting conformity against dissent,* is then iterated through many generational time steps.

Despite the complexity of its resulting segregative dynamics, the seceder model and its fluctuation-dominated behavior may best be understood from the perspective of critical phenomena. In this spirit, it is natural to consider the dimensionality dependences inherent to the model, with the expectation of finding a simpler, classical nonequilibrium dynamics within some sector of a larger parameter space. Wisdom in this regard may be had from the Bak-Sneppen model of punctuated evolution [13], wherein a like, innocuously trim update algorithm engendered extraordinarily rich spatiotemporal dynamics. Even so, a mean-field (MF) limit of this model was subsequently engineered [14,15] and further explored [16], with simplified scaling retrieved by introducing system-wide correlations to the interactions. Here, for the seceder model, we are motivated by similar goals. Clearly, the number of parameters is restricted—we have the population size N, understood to diverge in the thermodynamic limit. There is also the size m of the selection multiplet; finally, the dimensionality d of the genotype or ideology variable, which determines the nature of the base space through which various population clusters mark their trajectories, provides an additional degree of freedom.

We begin by enlarging the selection group from which the parent is chosen. Naively, we would expect the limit $m \to N$, which introduces increasing cross correlation within the society, to elicit, eventually, MF behavior, if only in the extreme case when $m = N$, when we are averaging over the entire population using the societal mean to determine the most distant, reproduced individual. Indeed, this is the case. The surprise, however, comes with the abruptness of the transition. There is already a marked change of behavior as we switch from triplet to quartet selection. In Fig. 1, we show single runs of the $d = 1$ seceder model using selection multiplets $m = 3$–8, within a large population, $N = 512$. For $m = 3$, we have trademark seceder demeanor, the branching characterized by three dominant but fluctuating arms, centered about the origin, with ample small-scale stochastic structure associated with the transient appearance of variously short-lived clusters. Rather than a gradual transition, we find for $m = 4$ that the typical stable configuration *suddenly* involves two branches, not three. In addition, these two branches exhibit only very modest fluctuations. Increasing the selection group to $m = 5$ further diminishes the noise, but hardly affects the tilt of what seems to be the nearly *linear* divergence of the two clusters. Next, for $m = 6$ and 7, there is a discrete jump to an altogether different, but closely allied, pair of trajectories. With $m = 8$ (and 9), another jump, and so it goes with each successive even-odd pair of selection multiplets. As $m \to N$, the trajectories are devoid of fluctuation, exhibiting classic MF behavior. Thus we see that the dominant dynamic of secession involves, for $m \geq 4$, segregation into two evenly populated opposing groups with a free interchange of individuals over the course of time. Ensemble averaging over many realizations, we have systematically studied the growth of this intergroup separation. Only for triplet selection, $m = 3$, do we find a *fractional* power-law divergence, with scaling index very close to 3/4, our measured value being 0.74 ± 0.01.

With the seceder model defined as above, the ideological space is a continuum. Clearly, discretizing the model alters no essential features. Indeed, much can be gleaned by considering this discrete model in its *deterministic* limit, wherein the most distinct individual is reproduced exactly, rather than yielding a merely approximate next of kin. One is led to a set of nonlinear coupled ordinary differential equations (ODEs) [12], first-order rate equations for the concentration simplex $(x_1, x_2, \ldots, x_B)$, describing the evolution of the discrete set of B genotypes possible within the population: $\dot{x}_j = \sum_{i_1,\ldots,i_m=1}^{B} \alpha^j_{i_1 i_2 \ldots i_m} \times x_{i_1} x_{i_2} \ldots x_{i_m} - x_j$, for $j \in (1, \ldots, B)$. These equations transform the seceder model into an evolving chemical reaction system whose dynamics are dictated by the law of mass kinetics and the constraints of unit dilution flux, possessing some features reminiscent of earlier efforts on generalized replicator equations [17]. Here one is looking at the stability of a B-branch solution generated by

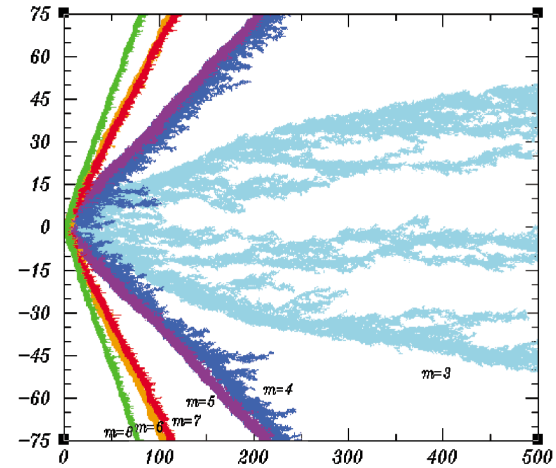

FIG. 1 (color). Space-time plots of the $d = 1$ stochastic seceder model with selection multiplets $m = 3$–8. For $m = 3$, several groups occur; however, for $m \geq m_c = 4$, the segregational dynamics yields just two repelling clusters, characterized by increasing homogeneity in the societal seceder limit.

m-multiplet selection group dynamics. The α coefficients are zero unless the genotype is the distant outlier—either in isolation, in which case $\alpha = 1$, or, as happens occasionally, in sharing that distinction with p other individuals: $\alpha = 1/p$. As a practical matter, these coefficients, combinatoric in origin, can be generated systematically via multinomial expansion $(x_1 + x_2 + \cdots x_B)^m$, carefully dividing numerical prefactors in appropriate proportions among relevant rate variables $\dot{x}_i$. Probability conservation demands this connection to the multinomial expansion, but it is the parceling out of terms that guarantees the complexity of the model. With this set of ODEs in hand, essentially providing a coarse-grained real space renormalization group (RG) prescription of the original seceder model, we follow the flow equations for the concentration variables, characterizing all relevant fixed points. Within this broader mB space, the $d = 1$ seceder model exhibits its full richness. As an indication of the wealth of this geometric pattern formation, consider the triplet selection dynamic, $m = 3$, where a self-similar hierarchy of multibranch fixed points emerges. We illustrate, e.g., the case $B = 4$, for which

$$\dot{x}_1 = x_1^3 + 3x_1(x_2^2 + x_3^2 + x_4^2) + 3x_1x_2x_3 + 6x_1x_3x_4 - x_1,$$

$$\dot{x}_2 = x_2^3 + 3x_2(x_1^2 + x_3^2 + x_4^2) + 3x_2x_3x_4 - x_2.$$

Because of branch symmetry and the normalization constraint $x_2 = 1/2 - x_1$, demanding $\dot{x}_1 = 0$, we are led to the cubic equation $7x_1^3 - 6x_1^2 + 5/4x_1 = 0$, yielding a unique *superstable* FP $(x_1, x_2, x_3, x_4) = (\frac{5}{14}, \frac{1}{7}, \frac{1}{7}, \frac{5}{14})$, as well as the less stable 2-branch solutions $(\frac{1}{2}, 0, 0, \frac{1}{2})$ and $(0, \frac{1}{2}, \frac{1}{2}, 0)$. The situation for $B = 5$ is slightly different—insisting upon symmetry $x_5 = x_1$ and $x_4 = x_2$, in addition to the normalization constraint $x_3 = 1 - 2x_2 - 2x_1$, we have the recurring 2-branch solutions with $x_1 = 0$ and $x_2 = 1/2$ and vice versa, leaving us with two coupled bilinear equations in the variables x_1 and x_2; graphically a rotated displaced ellipse and line within the unit square. There are two intersection points: one fully fledged superstable 5-branch solution, $(\frac{4}{13}, \frac{2}{13}, \frac{1}{13}, \frac{2}{13}, \frac{4}{13})$; the other, an unstable lower dimensional 4-branch solution, $(\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4})$. The latter solution can, thanks to the gap $x_3 = 0$, be understood, via coarse graining as literally self-similar to its 2-branch cousin $(\frac{1}{2}, 0, \frac{1}{2})$. This sort of hierarchical connection manifests itself regularly whenever we uncover vanishing x_i in the branch structure. The most compelling instance of this phenomenon appears when we search for a superstable 8-branch FP. In fact, there is none. The ODE flows converge on a peculiar 6-branch solution, $(\frac{11}{40}, \frac{1}{8}, 0, \frac{1}{10}, \frac{1}{10}, 0, \frac{1}{8}, \frac{11}{40})$, which is *exactly* self-similar to the strongly attractive 3-branch fixed point $(\frac{2}{5}, \frac{1}{5}, \frac{2}{5})$. Interestingly, this *lesser* 8-branch solution is distinct from the $B = 6$ FP, roughly (0.30, 0.12, 0.08, 0.08, 0.12, 0.30), easily shown to be irrational, as is the 7-branch and all those beyond 8. The 9-, 10-, and 12-branch FPs show no zeros, but such behavior becomes increasingly rare. The $B = 11$ solution also has two missing branches, $x_4 = x_8 = 0$, but with $x_1 \approx 0.271$, $x_2 \approx 0.089$, $x_3 \approx 0.044$, $x_5 \approx 0.047$, $x_6 \approx 0.100$, can be coarse grained to a broad 3-branch, though in this case only approximately self-similar to our dominant FP $(\frac{2}{5}, \frac{1}{5}, \frac{2}{5})$; likewise, the 15-branch, although 19- and 21-branch FPs show three gaps and a self-similarity to the 4-branch $(\frac{5}{14}, \frac{1}{7}, \frac{1}{7}, \frac{5}{14})$.

An additional payoff of this RG treatment of the seceder model is an explanation of the relative stability of 2- and 3-branch solutions for triplet ($m = 3$) and higher multiplet ($m \geq 4$) selection groups; recall Fig. 1. The essential dichotomy can be understood pictorially by following the flows of the $(m, B) = (3, 3)$ replicator system: see Fig. 2(a), where we show the relevant section of the [111] plane, $x_1 + x_2 + x_3 = 1$, our superstable fixed point, $(\frac{2}{5}, \frac{1}{5}, \frac{2}{5})$, within the unit triangle. We note, in particular, that the outermost 2-branch solution $(\frac{1}{2}, 0, \frac{1}{2})$ is *unstable* to small perturbations off the edge. In turn, the single branch FPs at the triangle vertices are entirely unstable. As $m \to 4$, however, the interior FP migrates to the midpoint of the triangle's lower edge, reversing the flows and stabilizing the 2-branch dynamic. From this vantage point, it is clear that quartet, rather than triplet, selection is the marginal case, a fact quickly confirmed by a stability analysis of the $(\frac{1}{2}, 0, \frac{1}{2})$ FP for arbitrary m. To linear order, we find $\dot{x}_2 = [2\binom{m}{1}(\frac{1}{2})^{m-1} - 1]x_2 + O(x_2^2)$, so we flow back to vanishing x_2 for $2m \leq 2^{m-1}$; i.e., $m \geq 4$, since the prefactor of the quadratic term is negative for the marginal value $m_c = 4$: see Fig. 2(b), which shows the full behavior. In fact, for a continuously variable polling group of size $m = 4 - \varepsilon$, the *perturbatively stable* 3-branch FP is located off the triangle's lower edge, with $x_2 = \frac{\varepsilon}{12}(\ln 2 - \frac{1}{4})$.

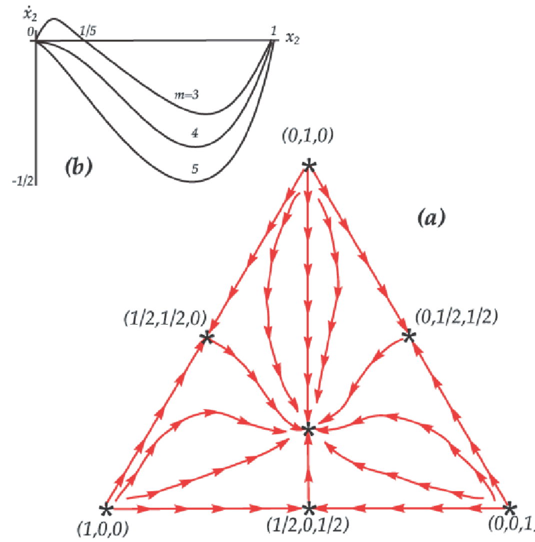

FIG. 2 (color online). (a) Flows in $x_1x_2x_3$ space for the $d = 1$ deterministic seceder model. A superstable 3-branch FP exists within the equilateral triangle for $m < 4$ only. (b) Middle branch growth rate, $\dot{x}_2$, for different multiplet selection group sizes, $m = 3$–5, revealing *stability* of the 2-branch solution $(\frac{1}{2}, 0, \frac{1}{2})$ for quartet selection and greater.

As one considers the stability of higher ($B > 3$) multibranch FPs, $m_c = 4$ remains the *upper critical size* of the selection subset; e.g., with $(m, B) = (4, 4)$, the superstable FP lies at the midpoint of the edge connecting the x_1 and x_4 vertices of the unit tetrahedron—that is, $(x_1, x_2, x_3, x_4) = (\frac{1}{2}, 0, 0, \frac{1}{2})$. All initial starting points within the tetrahedron flow outwards to the FP on this edge. While the other five edges of the tetrahedron are stable along their lengths, they are unstable in all other directions; the vertices, corresponding to 1-branch solutions, are maximally unstable. For triplet selection, our superstable 4-branch FP is $(\frac{5}{14}, \frac{1}{7}, \frac{1}{7}, \frac{5}{14})$, which lies within the tetrahedron. There are also stable 3-branch FPs in this case located in the faces of the tetrahedron, where one of the $x_i = 0$, but these are unstable to perturbations toward the interior. These findings might suggest, at least initially, that the greatest stability is associated with the largest number of branches, i.e., the $(B - 1)$-branch being unstable to B-branch solutions, etc. However, the absence of superstable 8-branch FP, for starters, and the subsequent appearance of gaps for $B = 11, 13, 17$ in the spectrum indicate that any runaway tendency toward proliferation of branches from tip splitting will be cut off. Indeed, that is precisely the characteristic behavior of the triplet seceder model, where 3-branch dynamics are typically seen, with occasional 4-, 5-, or 6-branch runs.

We should stress, in this regard, that the 3-branch solution is an extraordinarily robust feature of the model, becoming even more so in higher dimensions,

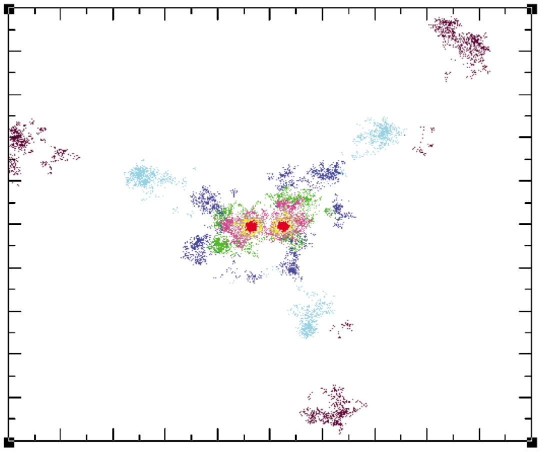

FIG. 3 (color). Despite greatly biased initial conditions, with two distinct localized groups, the $d = 2$ triplet seceder model evolves the population toward an endgame of three equidistant, separating clusters. Here $N = 2000$, and the temporal snapshots, indicated by different colors, correspond to successive times $t = 2^i$, for $i = 2$–8, red through mauve, respectively.

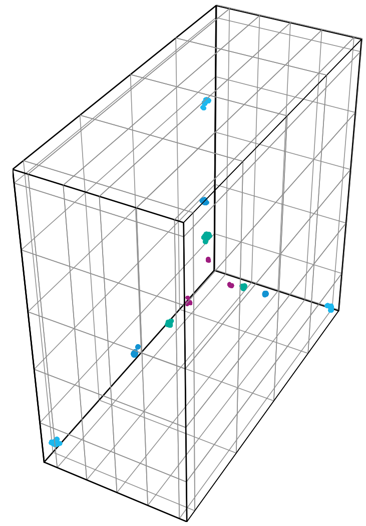

FIG. 4 (color). Self-organized secession in $d = 3$, as well as higher dimensions, collapses to the plane defined by three divergent groups. Population $N = 1000$; generations $t = 2^{9-12}$.

where the genotype is specified by a d-component vector rather than a scalar, the case we have focused on thus far. For example, in $d = 2$, an initially homogeneous, or highly polarized population for that matter (see Fig. 3), will eventually segregate into three distinct diverging groups. In $d = 3$ dimensions, we might expect four clusters, localized at the corners of an expanding tetrahedron, preserving the notion of equal distance; however, this does not happen at all. Again, we observe the formation of just three groups—note Fig. 4. The effect is stronger still for $d \geq 4$. Apparently, asymptotic higher-dimensional secession involves segregational collapse to a greatly reduced, 2D subspace—the hyperplane defined by a coarsened trio of condensed opinion clusters whose angular orientation may vary from one realization to the next, but whose essential geometry does not. Interestingly, this dimensional reduction can also be understood within the deterministic model—one considers the stability of $d + 1$ equally separated clusters in d dimensions to perturbations (ultimately, statistical in nature) that bring one group closer to the rest [18]. For $d \geq 3$, we find that the errant group is unstable and goes extinct, whereas for $d < 3$, the zero FP associated with this vanishing group reverses stability, yielding three clusters with relative populations set by the degree of symmetry breaking.

In summary, we have revealed the mean-field limit of the multidimensional seceder model. For selection multiplet sizes $m \geq m_c = 4$, the nonequilibrium dynamics produce a steady state with two opposing groups, independent of d. In the extreme societal seceder limit ($m \to N$), the noisy dynamics dies away, leaving two tightly knit clusters. For $m = 3$, multiple groups are typical, with three the norm. Higher-dimensional genotypes produce, surprisingly, no further fragmentation, rendering irrelevant excessive ideological degrees of freedom. Using a coarse-grained RG prescription, which discretizes and renders deterministic the model, we analytically uncover a self-similar hierarchy of multiple branch FPs in a gapped spectrum. Additional work, concerning kinetic symmetry breaking, population fragmentation, and party dynamics in an evolutionary political game, will be reported separately [18]. In a more speculative vein, we are presently examining the broader dynamical significance of $m_c = 4$, e.g., regarding the relative stability of two versus multiparty political systems.

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