Disorder-Driven Collapse of the Mobility Gap and Transition to an Insulator in the Fractional Quantum Hall Effect

D. N. Sheng,[1] Xin Wan,[2] E. H. Rezayi,[3] Kun Yang,[2] R. N. Bhatt,[4] and F. D. M. Haldane[5]

[1]*Department of Physics and Astronomy, California State University, Northridge, California 91330, USA*
[2]*National High Magnetic Field Laboratory and Department of Physics, Florida State University, Tallahasse, Florida 32306, USA*
[3]*Physics Department, California State University Los Angeles, Los Angeles, California 90032, USA*
[4]*Department of Electrical Engineering, Princeton University, Princeton, New Jersey 08544, USA*
[5]*Department of Physics, Jadwin Hall, Princeton University, Princeton, New Jersey 08544, USA*

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We study the $\nu = 1/3$ quantum Hall state in the presence of random disorder. We calculate the topologically invariant Chern number, which is the only quantity known at present to distinguish unambiguously between insulating and current carrying states in an interacting system. The mobility gap can be determined numerically this way and is found to agree with experimental value semiquantitatively. As the disorder strength increases towards a critical value, both the mobility gap and plateau width narrow continuously and ultimately collapse, leading to an insulating phase.

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Two-dimensional electron systems in a perpendicular magnetic field have been the focus of both theoretical and experimental attention for the past two decades. Such systems, if made sufficiently pure and taken to low enough temperatures, exhibit primarily the fractional quantum Hall effect (FQHE) [1–3]. While the pure systems, especially for the strongest FQHE, are fairly well understood, there is essentially no quantitative understanding of the role of random disorder. The importance of disorder, however, is underscored by the hallmark FQHE plateaus, which cannot occur in a translationally invariant system. On the other hand, when the disorder strength becomes comparable to the strength of interactions between electrons, the FQHE will eventually be destroyed. In this Letter, we report on finite-size studies of the effects of random disorder on the 1/3 FQHE.

The usual numerically calculated quantities such as the energy spectrum, wave functions, and the density-density correlation functions, while useful in studies of isolated impurities [4,5], provide little or no understanding of transport properties. A more appropriate approach is to obtain the quantum Hall conductance by calculating topologically invariant Chern integers [6–8] in systems with periodic boundary conditions (or torus geometry), in the presence of random disorder. Physically the Chern integers are the boundary condition averaged Hall conductance of the system in units of e^2/h. In the case of the integer quantum Hall effect (IQHE), the Chern numbers for the ground states have fixed nonzero integer values, while a "metallic" or critical state that separates two neighboring IQHE states gives an intrinsically fluctuating Chern number [8]. (It takes on different integer values in response to slight changes of the external parameters, such as the disorder configuration.) Such a quantum phase transition has been studied for only several *noninteracting* models [8], in which only the Chern number for single-particle states needs to be calculated.

The situation is very different in the case of FQHE. Because of the topological and many-body nature of the problem, in the FQHE system at filling factor $\nu = nh/eB = p/q$ (B is the magnetic field and n the areal density), there exists a manifold of q nearly degenerate low energy states on the torus, whose energy differences disappear in the thermodynamic limit (Ref. [9] and see below). The quantization of the Hall conductance cannot be tied, in a physically meaningful way, to any particular one of the q ground states. For example, in the absence of disorder (in which case the degeneracy is exact), when an external flux quanta is inserted adiabatically [6,10,11] in a region inaccessible to the electrons, the states within a given manifold evolve into each other. In the presence of disorder, the Chern number of the individual states fluctuates while the sum of the Chern number of all the states turns out to be p and robust. As a consequence, we may regard the total Chern number p to be shared by the q degenerate states, which results in fractionally quantized Hall conductance $\sigma_H = pe^2/qh$ for the system.

Based on this picture, we have developed a numerical method for studying the topological Chern number of the ground state and low energy excited states for an *interacting* system. We will show that the presence of disorder leads to several interesting and important results for 1/3 FQHE: (i) A weak random disorder lifts the degeneracy of the ground state for a finite number of electrons. However, the level spacings between the lowest three states decrease monotonically with the increase of electron number, indicating the recovery of the degeneracy in the thermodynamic limit. (ii) The mobility gap, which separates the higher energy extended excitations from the low energy FQHE states, can be determined from the distribution of

the Chern number of the many-body states. This is the only way to distinguish unambiguously between insulating and current carrying states in an interacting system. (iii) In general the mobility gap, determined this way, will be different from the spectrum gap (which separates the lowest three states from other higher states). It is the mobility gap that should be compared with the experimentally obtained activation energies. (iv) There exists a critical disorder strength $W = W_c \approx 0.2e^2/\epsilon$ for Gaussian white noise potential, which marks a transition from the FQHE to insulator. (v) The physics of the destruction of the FQHE can be described as the continuous collapse of the mobility gap; the closing of the mobility gap and the destruction of the Hall conductance quantization occur at the same time.

We consider a two-dimensional interacting electron system in an $L_1 \times L_2$ square cell with twisted boundary conditions: $T(\mathbf{L}_j)\Phi(\mathbf{r}) = e^{i\theta_j}\Phi(\mathbf{r})$, where $T(\mathbf{L}_j)$ is the magnetic translation operator and $j = 1, 2$ represents the x and y directions, respectively. Calculating the Hall conductance σ_H directly from a Kubo formula would require a knowledge of all the many-body eigenstates. This proves impractical for systems with three electrons or larger. As first realized by Thouless and co-workers [6], a topological property of the wave function [10], known as the first Chern number, can be used to calculate the boundary condition averaged σ_H. The importance of Chern numbers, however, goes beyond obtaining σ_H. It appears to be the only quantity that distinguishes between insulating and current carrying states [7] in an interacting system. Because we are dealing with many-body wave functions, other simpler numerical methods (such as inverse participation ratio and Thouless numbers) used for determining the localization of the single-particle wave functions have no obvious extension here. The boundary-phase averaged Hall conductance for the kth many-body eigenstate can be written as $\sigma_H^k = C(k)e^2/h$, where

$$C(k) = \frac{i}{4\pi}\oint d\theta_j \left\{ \left\langle \psi_k \left| \frac{\partial \psi_k}{\partial \theta_j} \right\rangle - \left\langle \frac{\partial \psi_k}{\partial \theta_j} \right| \psi_k \right\rangle \right\}, \quad (1)$$

and the closed path integral is along the boundary of a unit cell $0 \le \theta_1, \theta_2 \le 2\pi$ (summation over j is implied). $C(k)$ is exactly the Berry phase (in units of 2π) accumulated for such a state when the boundary phase evolves along the closed path. To determine the Chern number uniquely [8], we separate the boundary-phase space into approximately 36–100 mesh points and get the sum of the Berry phases from each mesh. In cases where there are near-level crossings, the integration contour has to be chosen reasonably close to these points. This determines the size of the mesh, at least locally. For the mesh sizes we chose, we found the Chern numbers had converged and did not change by further reducing the mesh size.

In the presence of a strong magnetic field, one can project the Hamiltonian onto the partially filled, lowest Landau level. The projected Hamiltonian in the presence of both Coulomb interaction and disorder [12] can be written as

$$H = \frac{1}{A}\sum_{i<j}\sum_{\mathbf{q}} e^{-q^2/2}V(q)e^{i\mathbf{q}\cdot(\mathbf{R}_i - \mathbf{R}_j)} + \sum_i \sum_{\mathbf{q}} e^{-q^2/4}U_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{R}_i}, \quad (2)$$

where $\mathbf{R}_i$ is the guiding center coordinate of the ith electron, $V(q) = 2\pi e^2/\epsilon q$ is the Coulomb potential, $U_{\mathbf{q}}$ is the impurity potential with the wave vector $\mathbf{q}$, and A is the area of the system. We set the magnetic length $\ell = 1$ and $e^2/\epsilon\ell = 1$ for convenience. The Gaussian white noise potential we use is generated according to the correlation relation in q-space $\langle U_{\mathbf{q}}U_{\mathbf{q}'}\rangle = (W^2/A)\delta_{\mathbf{q},-\mathbf{q}'}$, which corresponds to $\langle U(\mathbf{r})U(\mathbf{r}')\rangle = W^2\delta(\mathbf{r}-\mathbf{r}')$ in real space, where W is the strength of the disorder. We consider the case of $\nu = N_e/N_\phi = 1/3$, where N_e and N_ϕ are the number of electrons and flux quanta. We obtain the exact low energy eigenvalues and eigenstates using the Lanczos method for systems up to eight electrons. We then calculate the Chern numbers using Eq. (1).

Independent of boundary conditions, we find that, for weak disorder, low energy states are separated into groups. The lowest group has three closely spaced states. This property comes from the threefold center of mass degeneracy of the pure system at $\nu = 1/3$ and is shared by the entire spectrum. For finite sizes, there is a finite level spacing between the lowest three states, which is much smaller than the energy difference between the third and the fourth states. The latter is the low energy spectrum gap denoted as E_s. At $W = 0.06$, we have $E_s = 0.04 \pm 0.002$ (averaged over 100–2000 disorder configurations), which is size independent for $N_e = 5$–8. It remains finite for weak W until W is further increased to $W \approx 0.15$, at which point E_s becomes too small and its value in the thermodynamic limit cannot be extrapolated from the sizes accessible to our approach.

On the other hand, the level spacings between the lowest three states depend strongly on N_e. They are expected to vanish exponentially as the linear dimension of the system [9]. This can be seen clearly in Fig. 1, where we show a semilog plot of the "bandwidth" E_b vs $\sqrt{N_e}$ for three different disorder strengths. It is apparent that E_b drops to zero for large N_e, a direct consequence of the topological order in the ground state. This is in contrast to the usual effect of disorder in the IQHE, where the degeneracy of all the states in a Landau level, not being a topological property, will be lifted by the perturbation of arbitrarily weak disorder.

In the presence of weak disorder ($W < 0.12$), we find that the total Chern number carried by the lowest level is always 1, for $N_e = 3$–8 and thousands of disorder configurations. Note that the vanishing E_b and the existence of a finite spectrum gap E_s, at relatively weak W, is a direct manifestation of the $\nu = 1/3$ FQHE; since the lowest three states become degenerate in the thermodynamic

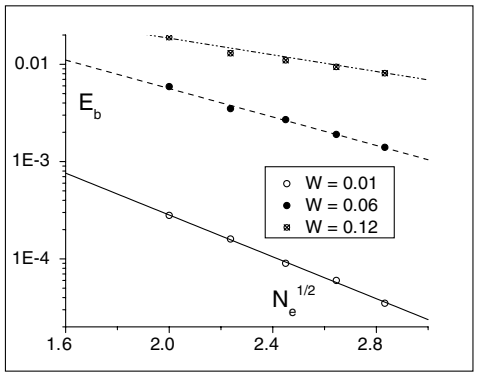

FIG. 1. A semilog plot of the bandwidth E_b of the first level that consists of three closely spaced lowest energy states as a function of the square root of the electron number, at three relatively weak disorder strengths $W = 0.01$, 0.06, and 0.12.

limit, the Hall conductance carried by the ground state is $1/3$ of e^2/h. However, this is not unique to the ground state, and we find that even the low lying excited states above the spectrum gap have similar properties. Namely, each three states carry total Chern number $C = 1$. At $W = 0.01$, the lowest two groups have $C = 1$ for all the disorder configurations we have sampled. In Fig. 2 we plot for the first five groups, $P(C)$, for $W = 0.06$, vs C. $P(C)$ is the probability that the total group Chern number is C. We have averaged over 2000, 1000, and 500 disorder configurations for $N_e = 5, 6$, and 7, respectively. As seen in Fig. 2, $P(1) = 1$ for the first group ($N_g = 1$), and $P(C) = 0$ for $C \neq 1$. This means all disorder configurations have $C = 1$, which corresponds to the $1/3$ FQHE because each state carries a definite average Hall conductance of $e^2/3h$. For $N_g = 2$, both $P(0)$ and $P(2)$ are nonzero, indicating that a small number of configurations have $C = 0$ or 2. As a result, $P(1)$ is reduced to 0.9, 0.91, and 0.92 for $N_e = 5$, 6, and 7. The increase of $P(1)$ with N_e indicates that $P(1)$ may recover to 1 at large N_e. For the $N_g = 3$ case, $P(1)$ is significantly reduced to about 0.7; it behaves nonmonotonically as a function of N_e. This results from the coexistence of three different Chern numbers in the thermodynamic limit, which characterizes the delocalization of quasiparticle excitations. Namely, these excitations carry nonzero Chern numbers which are extended in real space. For $N_g > 3$, $P(1)$ is further reduced and seems to saturate at a value near 0.5.

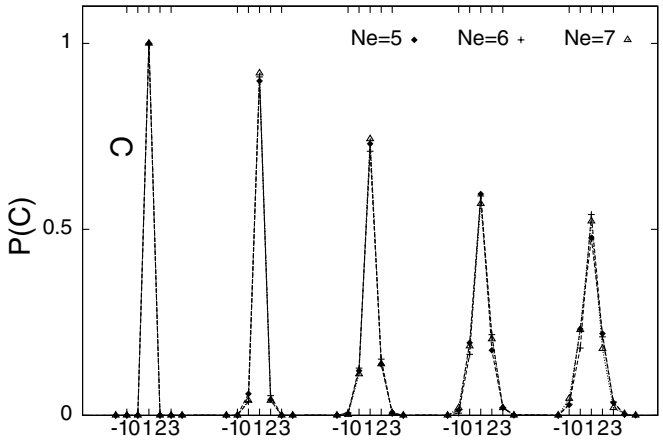

FIG. 2. The probability distribution $P(C)$ of total Chern number C, for the lowest five groups of states. The energy of the states increases towards the right.

We may regard the fluctuation of the Chern number to be an indication of the degree of delocalization. In analogy to the physics of noninteracting systems [8], we define $P_{\text{ext}} = 1 - P(1)$ to be the likelihood of the breakdown of the quantization of the Hall conductance and thus a measure of the delocalization of the charged excitations. In the FQHE plateau regime, P_{ext} goes to zero as a result of the localization or nondissipative nature of the state. Here nonzero P_{ext} occurs as we go to higher energy states ($N_g \geq 3$). The energy that separates these two kinds of states is called the mobility edge, where P_{ext} has a large increase, which probably becomes a finite jump in the thermodynamic limit. For $W = 0.06$, we find that, from $N_g = 2$ to 3, P_{ext} has the largest increase, which puts the $N_g = 3$ group at the mobility edge. Measuring the energy at the mobility edge relative to the energy of the lowest level, we get the mobility gap E_m for $N_e = 4$–8, which is N_e dependent. For $W \geq 0.06$, we determine the mobility gap by extrapolating the finite-size data to the thermodynamic limit. For weaker disorder, the sizes that can be treated are not sufficient to produce a meaningful extrapolation to $N \rightarrow \infty$.

The extrapolated E_m vs W is shown in Fig. 3(a). In the inset we plot E_m vs $1/N_e$ for $W = 0.17$, which can be best fit to $E_m = 0.10/N_e + 0.005$; thus we obtain $E_m = 0.005 \pm 0.003$ in the thermodynamic limit. We see that at $W \approx 0.17$ such a gap is strongly reduced, consistent with the drop of the spectrum gap for similar W, signaling the FQHE is on the verge of being destroyed by disorder.

The energy gap Δ in the excitation spectrum of the correlated many-body ground state can be extracted experimentally from the temperature dependence of the magnetoresistivity, $\rho_{xx} \propto \exp(-\Delta/2k_BT)$, where $\Delta/2$ is the activation energy and k_B the Boltzmann's constant [13,14]. Boebinger *et al.* [13] systematically studied the activation energy for $\nu = 1/3$, $2/3$, $4/3$, and $5/3$ and its dependence on sample mobility μ (an indication of disorder) in GaAs-$Al_xGa_{1-x}As$. For a class of high-mobility (at that time) samples, they found that $\Delta \simeq 0.049e^2/\epsilon l - 6$ K. The mobility (μ) dependence of Δ can then be extracted from the known dependence of μ on the electron density n of these samples, since n determines the magnetic field B (or l) at the $1/3$ family of fillings. For semiquantitative comparison, we use a typical dependence, $\mu = \mu_0(n/n_0)^{1.5}$, where $\mu_0 = 600\,000$ cm^2/V s and $n_0 = 1.5 \times 10^{11}$ cm^{-2}, as extracted from Fig. 1 of Ref. [13]. Figure 3(b) compares this empirical formula of $\Delta(\mu)$ with the mobility gap we obtained in our calculation. In the latter, we assume that both the (zero field)

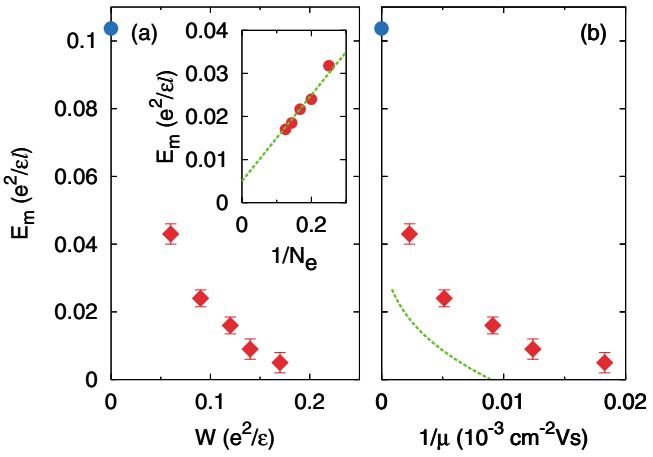

FIG. 3 (color). (a) The extrapolated mobility gap E_m as a function of W. Inset shows E_m at $W = 0.17$ for $N_e = 4$–8 electrons. The dashed line indicates E_m can be extrapolated to 0.005 at $1/N_e \to 0$. The blue dot at $W = 0$ is the creation energy for a quasiparticle-quasihole pair at infinite separation, extrapolated from pure systems with up to $N_e = 10$. (b) Dependence of E_m on mobility μ. The dashed line is converted from a fit to experimental data (taken from Ref. [13]). Here, we use an empirical mobility-density relation as well as a mobility-disorder relation in the Born approximation (see text for detail).

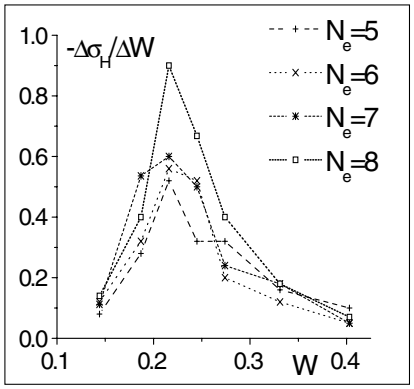

FIG. 4. The relative decrease of the Hall conductance of the lowest level over the change of disorder strength ΔW is plotted as a function of W for $N_e = 5$, 6, 7, and 8.

mobility and the (high field) mobility gap are dominated by short-range scatterers (appropriate for these so-called high-mobility samples) and, in the Born approximation, $\mu = e\hbar^3/(m^{*2}W^2)$. Here, we do not include the effects of layer thickness and Landau level mixing which, nevertheless, exist in experimental samples and are known to reduce the gap by as much as a factor of 2 [14,15] (for zero or very weak disorder).

As we further increase W, the FQHE becomes unstable. This can be discerned by following the evolution of the Chern number of the lowest level and σ_H averaged over the lowest three states. For example, for $N_e = 6$, at $W \leq 0.14$, we have $\sigma_H = e^2/3h$; it drops to $0.924e^2/3h$ at $W = 0.17$. At larger W, we find a very strong enhancement of the fluctuation in the Chern number and, correspondingly, a rapid reduction of σ_H. Similar results are obtained for $N_e = 8$. As shown in Fig. 4, $-\Delta\sigma_H/\Delta W$ has its largest value near $W_c = 0.22 \pm 0.025$ for all $N_e = 5$–8, which determines the critical disorder for the $\nu = 1/3$ state plateau to insulator transition.

We have also studied the gap for filling factors slightly away from 1/3. At weak W, the $\nu = 1/3$ FQHE plateau has a finite width due to the nonzero mobility gap that survives to fillings slightly below and above 1/3. The width of the plateau and the mobility gap at 1/3 both decrease with the increase of the disorder strength, and they are expected to vanish at the same disorder strength. Our results are consistent with this expectation.

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