Information Flow in Chaos Synchronization: Fundamental Tradeoffs in Precision, Delay, and Anticipation

Shawn D. Pethel,* Ned J. Corron, Quitisha R. Underwood, and Krishna Myneni

U.S. Army Aviation and Missile Command, AMSAM-RD-WS-ST, Redstone Arsenal, Alabama 35898, USA

(Received 17 October 2002; published 23 June 2003)

We use symbolic dynamics to examine the flow of information in unidirectionally coupled chaotic oscillators exhibiting synchronization. The theory of symbolic dynamics reduces chaos to a shift map that acts on a discrete set of symbols, each of which contains information about the system state. Using this transformation we explore so-called achronal synchronization, in which the response lags or leads the drive by a fixed amount of time. We find fundamental tradeoffs between the precision to which the drive state is detected, the quality of synchronization attained, and the delay or anticipation exhibited by the response system. To illustrate these tradeoffs, we provide a physical example using electronic circuits.

DOI: 10.1103/PhysRevLett.90.254101 PACS numbers: 05.45.Xt, 05.45.Gg

Separate chaotic systems can synchronize if allowed to share information over a coupling channel [1]. The implications of this phenomenon have stimulated a great amount of research in both scientific and engineering disciplines, yet many issues remain concerning the properties and robustness of synchronization. Central to these issues is the fact that chaotic oscillators have nonzero Shannon entropy—that is, they act like information sources [2]. For unidirectional synchronization to occur this information must be encoded at the drive, transmitted through a coupling channel, and decoded at the response system. These processes are identical to those considered by modern communication theory [3]. It follows that communication concepts of information flow can be applied to the phenomenon of chaos synchronization provided an appropriate transformation is made.

Recently, Stojanovski *et al.* [4] used symbolic dynamics to determine the minimum channel capacity required to sustain chaotic synchronization in unidirectionally coupled systems. They showed that a channel capacity exceeding the Kolmogorov-Sinai entropy of the drive system is theoretically necessary and sufficient to sustain synchronization to any arbitrarily small error. In this Letter we continue the analysis of synchronization using this formalism. We find that a symbolic representation reveals a relationship between the quality of synchronization, the precision of the detection process at the drive, and the time delay or advance of the response with respect to the drive. We show that a delay between drive and response can be used to compensate for a poor detector at the drive; conversely, a very good detector can be exploited to allow anticipation in the response. These results are not system specific but are general properties of unidirectional synchronization of chaotic systems. As a physical example, we devise an experiment with two chaotic electronic circuits that explicitly demonstrates these effects.

To begin, we consider two identical chaotic oscillators, labeled drive (D) and response (R), which are connected by a unidirectional coupling channel. As shown in Fig. 1, D is coupled to a channel input by means of a detector, and the channel output is connected to R by a controller. The detector output can be a function of some or all of the dynamic variables of D, and the controller provides a perturbation to R according to the signal received via the channel. We say the two oscillators are synchronized when all dynamical variables of R follow the corresponding dynamical variables of D to within some specified fidelity. Typically, one compares the states of D and R at the same time; however, we also consider so-called *achronal synchronization*, in which the R lags or leads D by a fixed amount of time [5].

To provide an intuitive understanding of the information transmission required to sustain synchronization, we consider the drive system

$$\dot{x}_D = f(x_D), \qquad (1)$$

where x_D is a vector of the drive system states and f defines a flow. The response is

$$\dot{x}_R = f(x_R) + p(x_D, x_R), \qquad (2)$$

where x_R is the corresponding response states and p represents a controller designed to yield synchronization. Any effects due to the detector and channel are also included in p. We assume both oscillators can be represented as a return map through the method of Poincaré, and we denote the time of the ith return as t_i. Accordingly, D is described by a return map of the form

$$x_D(t_{i+1}) = F(x_D(t_i)), \qquad (3)$$

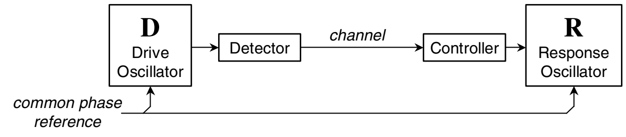

FIG. 1. Drive-response configuration for investigating synchronization with unidirectional coupling.

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and R can be described similarly; however, the controller p introduces additional terms that can be used to maintain synchronization at R.

We suppose D and R are synchronized at t_i with quality N_b, where N_b is the number of bits of precision. Given this initial condition and the flow f, the response system can predict the next return $x_D(t_{i+1})$ with a precision slightly less than N_b bits: the loss in precision is due to the divergent flow inherent to chaotic dynamics. The precise method by which R can make this prediction relies on an ingenious choice of the controller p — here it is sufficient to assume such a controller exists. Therefore, to maintain synchronization, system D need only send the bits required to make the predicted value of $x_D(t_{i+1})$ fully N_b bits accurate. The rate at which this information must be sent is simply the Kolmogorov-Sinai entropy of the system, and the theory of symbolic dynamics provides a transformation that allows us to isolate the information corresponding to these last bits of precision [4].

Symbolic dynamics is used to represent the continuous trajectory of a chaotic oscillator using discrete samples with finite precision [6]. To generate a symbolic representation of a trajectory, the state space containing the chaotic attractor is partitioned into regions, each of which is labeled with a symbol such as "0", "1", etc. Whenever the state enters a new region the corresponding symbol is generated; for flows represented by a return map, a new symbol is generated for each return. In this way each trajectory is mapped to a bi-infinite symbol sequence $\mathbf{s}$, such as "... 11101.11010111 ...", where the period indicates the current symbol in the sequence, future symbols are written toward the right, and past symbols to the left. Every trajectory on a chaotic attractor produces a unique symbol sequence. Likewise, every point x on a Poincaré surface can be mapped into sequence space by considering all of its images and preimages and replacing them with the appropriate symbol. We denote this mapping as $\mathbf{s} = r(x)$.

State space and sequence space are topologically equivalent provided the mapping r between them is one-to-one and continuous. That is, every observed system state corresponds to a unique symbol sequence and nearby points in state space are also nearby in sequence space. For two sequences $\mathbf{s} = \ldots s_{-2}s_{-1} \cdot s_0 s_1 s_2 \ldots$ and $\mathbf{t} = \ldots t_{-2}t_{-1} \cdot t_0 t_1 t_2 \ldots$ we define the distance between them by [7]

$$d(\mathbf{s}, \mathbf{t}) = \frac{1}{2} \sum_{i=-\infty}^{\infty} \frac{\delta(s_i, t_i)}{2^{-|i|}}, \tag{4}$$

where

$$\delta(s_i, t_i) = \begin{cases} 0, & s_i = t_i, \\ 1, & s_i \neq t_i. \end{cases} \tag{5}$$

Thus sequences $\mathbf{s}$ and $\mathbf{t}$ are nearby if they agree in the first few symbols about $i = 0$. A partition that guarantees these properties is *generating*, and the resulting dynamics in sequence space are entirely consistent with those in state space. Finding a generating partition is straightforward for one-dimensional mappings, but much less so in two or more dimensions [8]. Recently, efficient numerical algorithms have been designed for the general case [9,10]. In the following discussion we merely assume that such a partition exists.

An important aspect of the symbolic description is that it converts the dynamical system into a trivial shift map. In sequence space the action of the return map (3) is to shift all symbols one space to the left. If $\mathbf{s}_i = r(x_i)$, then $\mathbf{s}_{i+1} = (F(x_i))$ is computed in sequence space by simply moving the period one symbol to the right. In practice, we observe only a truncated representation of $\mathbf{s}_i$ consistent with our detector resolution. The shifting process introduces new information that appears on the right-hand side of the truncated sequence. To illustrate, we consider a finite symbolic description of the $\mathbf{s}_i$ such as "11101.11010111". The next return $\mathbf{s}_{i+1}$ will be represented as "11011.1010111?", where "?" is either a "0" or a "1". One can think of this symbol as a previously undetectable detail of the initial condition or, in an experimental setting, as ultimately originating from noise. This new symbol, then, is the information required to maintain synchronization.

With regard to synchronization, the symbolic dynamics representation immediately shows that *the channel need only transmit one symbol per cycle to maintain synchronization between D and R to any level of precision.* This is because only one new symbol is generated per return; all the other symbols describing the state were generated previously and have already been transmitted. On average, this new symbol contains H bits of information, where H is the Shannon entropy of the system. Synchronization cannot be maintained if less than this amount of information is transmitted per cycle. In this context, H is related to the Kolmogorov-Sinai entropy in bits per second by including the average symbol rate [3]. In the rest of this Letter we discuss factors that influence synchronization quality once this minimum condition is met.

To assess synchronization quality in sequence space, we let $\mathbf{s}$ and $\mathbf{t}$ denote the current symbolic state of D and R, respectively. We say $\mathbf{s}$ and $\mathbf{t}$ agree to m symbols when $d(\mathbf{s}, \mathbf{t}) \leq 2^{-m}$. As such, m quantifies the synchronization quality; topological equivalence guarantees that D and R rapidly approach each other in both sequence space and state space as m is increased. On average, R contains $Q_S = mH$ bits of information about D. Synchronization of this quality is maintained when the controller produces a perturbation such that the mth future symbol of D appears in the mth position of R. The cost of increasing m appears in the detector resolution: to detect the mth symbol the detector must be able to extract $Q_D = mH$ bits of information about the drive state. For true

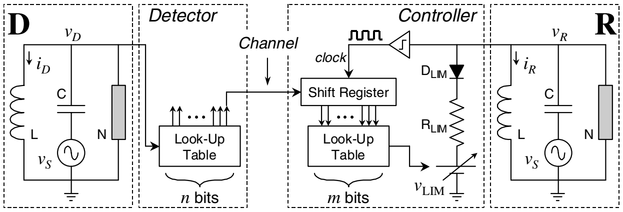

FIG. 2. Experimental configuration for demonstrating synchronization via a symbol channel. Oscillators D and R are identical chaotic oscillators, v_S is a low-level sine wave to phase lock the circuits, N represents nonlinear and active components, and $D_{\rm LIM}$ is an ideal diode limiter implemented using an active circuit.

synchronization R cannot contain more information than the detector is capable of extracting from D; thus, $Q_S \leq Q_D$.

In general, we can consider the case where we detect and transmit the nth symbol in D, while we control at the mth symbol in R. Accordingly, the resulting D and R waveforms will be time shifted by $L = m - n$ cycles. For $m > n$ the response lags the drive — a situation referred to as lag synchronization. In this case the quality of synchronization exceeds the detector resolution. The opposite scenario can be created as well. For $n > m$ the lag is negative and the response leads the drive. This effect has been seen previously and has been described as *anticipating* synchronization [11]. In sequence space we can clearly see that an upper bound on synchronization quality is $m \leq n + L$. Multiplying through by H, we get the equivalent statement in an information sense:

$$Q_S \leq Q_D + LH. \tag{6}$$

This bound on synchronization quality, which is valid for identical oscillators connected by a unidirectional and noiseless channel, illuminates a fundamental tradeoff between detector precision and achronal synchronization. Namely, synchronization quality can be enhanced by lag, even to the point that it can exceed the detector precision.

We now present a simple physical system in which the above bound can be explored. The configuration of the experiment is shown in Fig. 2. Systems D and R are piecewise-linear LC circuits of a type used and documented previously [12]. These oscillators exhibit a simply folded band attractor as shown in Fig. 3, with a dominant spectral peak near 1.2 kHz. To eliminate phase drift between the two circuits, a 30-mV (peak-to-peak) sine wave at 1.19 kHz is connected in series with the tank capacitor in each circuit. This low-level signal entrains the phase of both oscillators to a common reference but has a negligible effect on the amplitudes, which remain chaotic and uncorrelated.

Return maps constructed using successive waveform peaks for both D and R are shown in Fig. 3. The data indicate that the systems are well approximated by a one-dimensional, unimodal return map. Such maps have well-

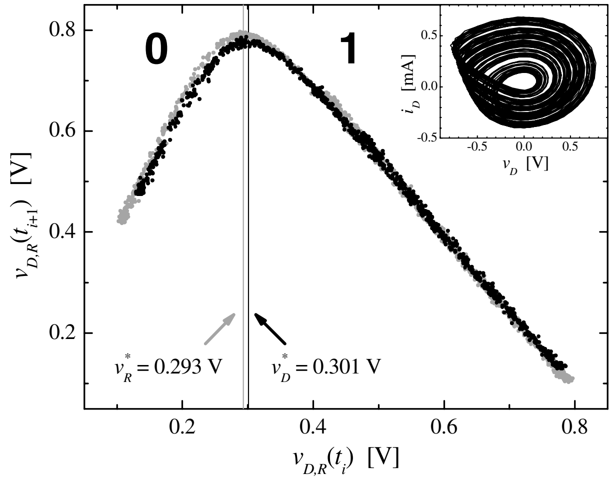

FIG. 3. Return map of successive peak voltages for D (black) and R (gray), including generating partitions (vertical lines). The inset shows the attractor for D.

understood properties in symbolic dynamics — namely, a two-symbol alphabet, a generating partition at the critical point, and no memory of past symbols. We have estimated the partition voltages to be $v_D^* = 0.301$ V and $v_R^* = 0.293$ V for D and R, respectively. Following the usual

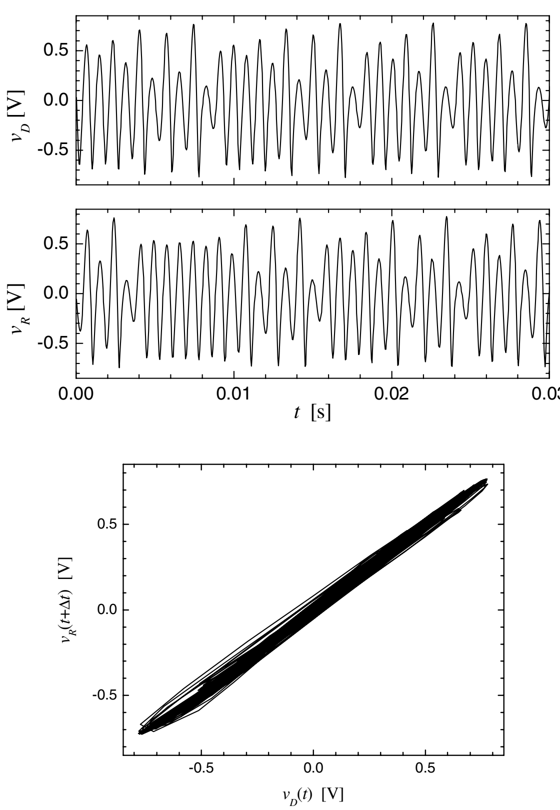

FIG. 4. Drive (v_D) and response (v_R) waveforms showing synchronization with an 8-cycle lag ($\Delta t = 6.7$ ms). D is detected with $n = 1$ bit of precision, and R is controlled at the $m = 8$th future bit.

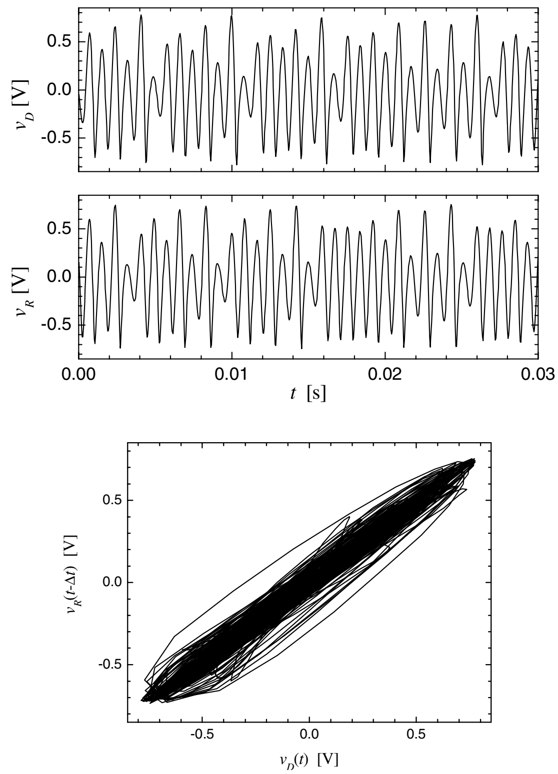

FIG. 5. Drive (v_D) and response (v_R) waveforms showing synchronization with 2-cycle anticipation ($\Delta t = 1.7$ ms). D is detected with $n = 9$, and R is controlled at $m = 6$.

convention, we say each oscillator generates the symbol "0" for a peak less than the partition voltage and the symbol "1" otherwise.

The lookup tables and shift register shown in Fig. 2 are implemented using a DSP card (Innovative Integration ADC64) hosted in a PC. The D waveform is sampled continuously by the DSP card at $\sim$40 kHz and a voltage peak is found for each cycle. The voltage peak is converted by a lookup table into a symbol sequence, from which the nth symbol is detected and transmitted to R. At R, the new symbol is shifted into the least significant bit of an m-bit shift register that is clocked at the symbol arrival rate. The symbol sequence held in the shift register is mapped by the lookup table back into a voltage representing the state of D: we use this voltage to control R via dynamic limiting [13]. We allow one cycle of oscillator time for this processing. The lookup tables, which serve as the mapping $r(x)$ for D and R, are assembled beforehand by observing each uncontrolled oscillator for 200 000 cycles and averaging initial conditions for each observed sequence [12]. We note that any chaos control technique for encoding symbolic dynamics can be used with equal effect [14–17].

With this setup we can vary n and m as discussed earlier. In Fig. 4 we show results for the case $n = 1$ and $m = 8$, corresponding to a low-precision detector and a high-precision controller. As shown, the controller forces R to accurately reconstruct the D waveform, albeit with a lag time of eight cycles (m–n, plus one processing cycle). The large lag thus enables relatively high quality achronal synchronization despite detecting D with only 1-bit precision. For the results in Fig. 5 we use $n = 9$ and $m = 6$, so that the precision of the detector exceeds that of the controller. Again, achronal synchronization is observed; however, R is now leading D by two cycles. This example of anticipating synchronization is a direct result of the high-precision available to the detector.

This experiment was designed to explicitly illustrate properties derived from the symbolic dynamical model of synchronization; in principle, all unidirectional chaos synchronization can be viewed in this way. The novelty of our experimental setup is in the sophisticated detector and controller that allow us to approach fundamental limits. In this sense we have constructed an ideal scenario. However, it is hoped that this extreme can be compared to more common scenarios with less contrived couplings, thereby providing an estimate of the overhead available to improve synchronization and the trade-offs necessary to achieve it.

*Electronic address: shawn.pethel@ws.redstone.army.mil

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