Resonant Tunneling between Luttinger Liquids: A Solvable Case

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We discuss the conductance of a Luttinger liquid interrupted by a quantum dot containing a single resonant level. Using bosonization and refermionization methods, we find a mapping to a Kondo-type problem which possesses a nontrivial Toulouse-type solvable point. At this point, we obtain an analytic expression for the nonlinear current-voltage characteristics and analyze the differential conductance and the width of the resonance peak as functions of bias and gate voltages, temperature, and barrier asymmetry. We also determine the exact scaling function for the linear conductance.

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The field of one-dimensional interacting metallic systems recently experienced another revival as single-wall carbon nanotubes (SWNTs) have been found to display transport properties consistent with the Luttinger liquid (LL) theory [1]. While the electrical transport through clean SWNTs has been investigated in different independent experiments, the transport properties of SWNTs with impurities (or in more complicated setups) are still to be studied in detail. Progress was recently made in this direction: In [2], the manufacture of quantum dots on the nanotube basis was reported. Surprisingly, the authors found that the transport is dominated by a coherent transmission (or resonant tunneling) in a wide parameter range.

The presence of a resonant level is known to enhance conductance. Indeed, for noninteracting electrons the local level hybridizes with the conduction band causing a Lorentzian shaped peak in the density of states (the conductance being related to the Breit-Wigner scattering cross section via the Landauer formula). Unless the system is exactly at resonance, the picture remains qualitatively the same for the case of interacting electrons, even LLs [3]. Therefore one expects the conductance to increase upon lowering the temperature. On the other hand, at low temperatures, the conductance is known to vanish (unless exactly at resonance) due to the effective enhancement of backscattering processes specific for the LLs. Hence, a nonmonotonic behavior of the linear conductance as a function of temperature. The limiting cases have been thoroughly studied in Refs. [3–6], but the full description of the crossover remains an open problem.

Recently, Nazarov and Glazman (NG) calculated the crossover conductance in the weak electron-electron interaction limit (when the LL parameter g is close to 1) by using the Landauer-type approach supplemented by renormalization group [4]. In this Letter, we wish to discuss the opposite limit of strong interactions, which is relevant for such systems as SWNTs. We shall concentrate on the special value of the coupling $g = 1/2$ and present an explicit solution of the problem at this point. At this particular value of the LL parameter and when the resonant level energy is tuned to match the equilibrium chemical potentials in the leads, the resonant tunneling process is marginally relevant and its amplitude increases logarithmically upon lowering the energy scale [3,7]. It turns out that, at low temperatures in the linear regime (i.e., in the limit of small bias voltage), the sequential tunneling dominates the transport for $g < 1/2$, while above that value the resonant transmission wins over [3,5]. Hence, apart from being supplementary to NG results, the exact solution at $g = 1/2$ yields insights into the interplay between these two transport mechanisms.

We model the system by a resonant level (which can also be regarded as a single state quantum dot, so we use both terms) coupled to interacting leads, which is described by the following Hamiltonian (we ignore the spin degrees of freedom throughout the Letter):

$$H = H_K + H_t + H_C, \qquad (1)$$

where H_K is the kinetic part, $H_K = \Delta d^\dagger d + \sum_{i=R,L} H_0[\psi_i]$, describing the electronic degrees of freedom in the leads $H_0[\psi_i]$, and the resonant level with energy Δ with the corresponding electron operators being $d^\dagger, d$. The dot can be populated from either of the two leads ($i = R, L$) via electron tunneling with amplitudes γ_i, $H_t = \sum_i \gamma_i [d^\dagger \psi_i(0) + \text{H.c.}]$. Here H_C describes the electrostatic Coulomb interaction between the leads and the dot, $H_C = \lambda_C d^\dagger d \sum_i \psi_i^\dagger(0)\psi_i(0)$. This interaction is a new ingredient we have introduced, absent in [3,4]. It does not, however, affect the universality as we shall show. The contacting electrodes are supposed to be one-dimensional half-infinite electron systems. We model them by chiral fermions living in an infinite system: The negative half-axis then describes the particles moving towards the boundary, while the positive half-axis carries electrons moving away from the end of the system. In the bosonic representation $H_0[\psi_i]$ are diagonal even in the presence of interactions (for a recent review see, e.g., [8]; we set the renormalized Fermi velocity $v = v_F/g = 1$, the bare velocity being v_F):

$H_0[\psi_i] = (4\pi)^{-1}\int dx[\partial_x\phi_i(x)]^2$. Here the phase fields $\phi_i(x)$ describe the slow varying spatial component of the electron density (plasmons), $\psi_i^\dagger(x)\psi_i(x) = \partial_x\phi_i(x)/2\pi\sqrt{g}$. The electron field operator at the boundary is given by [9], $\psi_i(0) = e^{i\phi_i(0)/\sqrt{g}}/\sqrt{2\pi a_0}$, where a_0 is the lattice constant of the underlying lattice model. Here g is the LL parameter (coupling constant) [3,8]. In the chiral formulation, the bias voltage amounts to a difference in the densities of the incoming particles in both channels far away from the constriction [10]. The current is then proportional to the difference between the densities of incoming and outgoing particles within each channel.

To the best of our knowledge, Hamiltonian (1) cannot be solved exactly even in the $g = 1$ case as long as λ_C remains finite. However, after a transformation of $d^\dagger$ and d operators to the spin representation of the form $S_x = (d^\dagger + d)/2$, $S_y = -i(d^\dagger - d)/2$, $S_z = d^\dagger d - 1/2$, one immediately observes that the λ_C term is analogous to the S_z-spin density coupling in the Kondo problem. The latter is known to be explicitly solvable at a particular value of the longitudinal coupling: the Toulouse limit (see, e.g., [8]). Let us perform a similar calculation. As a first step, we introduce new symmetric and antisymmetric fields $\phi_\pm = (\phi_L \pm \phi_R)/\sqrt{2}$. Then we apply the transformation $H' = U^\dagger H U$ with $U = \exp[iS_z\phi_+(0)/\sqrt{2g}]$ [11], which changes the kinetic and the Coulomb coupling parts of the full Hamiltonian to [we drop a constant contribution proportional to $S_z\delta(x)$ which is incorporated into a renormalization of Δ] $H_K' + H_C' = H_K + (\lambda_C/\pi\sqrt{2g} - \sqrt{2/g})S_z\partial_x\phi_+(0)$, and the tunneling part (terms containing γ_i) to

$$H_t' = (2\pi a_0)^{-1/2}[S_+(\gamma_L e^{i\phi_-/\sqrt{2g}} + \gamma_R e^{-i\phi_-/\sqrt{2g}}) + (\gamma_L e^{-i\phi_-/\sqrt{2g}} + \gamma_R e^{i\phi_-/\sqrt{2g}})S_-], \tag{2}$$

where $S_\pm = S_x \pm iS_y = d^\dagger, d$. At the point $g = 1/2$, one can refermionize the problem by defining new operators,

$$\psi_\pm = e^{i\phi_\pm}/\sqrt{2\pi a_0}, \tag{3}$$

which fulfill standard fermionic commutation relations. With the help of the particle density operator $\psi_\pm^\dagger\psi_\pm = \partial_x\phi_\pm/2\pi$, we can immediately write down the refermionized Hamiltonian,

$$H = H_0[\psi_\pm] + (\lambda_C - 2\pi)2S_z\psi_+^\dagger\psi_+ + \Delta S_z + S_+(\gamma_L\psi_- + \gamma_R\psi_-^\dagger) + (\gamma_L\psi_-^\dagger + \gamma_R\psi_-)S_-. \tag{4}$$

In the case of the symmetric coupling $\gamma_L = \gamma_R$, this Hamiltonian is similar to that of the two-channel Kondo problem and, at the Toulouse point $\lambda_C = 2\pi$, can be solved exactly (out of equilibrium) using the method of Ref. [12]. The novel ingredient in the following analysis is the extension to the asymmetric case. To take advantage of the Toulouse point, we set the Coulomb coupling amplitude to 2π in what follows. This not only removes the four fermion interaction but decouples the "$\pm$" channels making the "+' channel free.

As we already mentioned, due to the linear dispersion relation, the current through the system is proportional to the difference between the densities of particles moving towards the dot and away from it in either of the channels. Because of the chiral geometry, we then have $I \sim \psi_L^\dagger\psi_L(-\infty) - \psi_L^\dagger\psi_L(\infty)$, which, being transformed to "$\pm$" channels, results in $I \sim \psi_-^\dagger\psi_-(-\infty) - \psi_-^\dagger\psi_-(\infty)$. Since the "+" channel is free, it does not contribute to the above formula. As the "$-$" channel is also free when away from the dot, in order to calculate the current we need only to know the scattering matrix of "$-$" fermions determined by Hamiltonian (4). The chemical potential of the incoming particles is determined by the bias voltage. Hence, the current is given by (we set $e = 1$)

$$I(V) = G_0\int d\omega\, T(\omega)[n_F(\omega - V) - n_F(\omega)], \tag{5}$$

where n_F denotes the Fermi distribution function and $1 - T(\omega)$ is the energy dependent penetration coefficient of the "$-$" particles from $x < 0$ to $x > 0$. The prefactor $G_0 = e^2/h$ is fixed by the requirement that at perfect transmission one obtains the correct conductance.

The easiest way to obtain the transmission coefficient is the equations of motion method. Since we have two types of operators, for the electrons of the "$-$" channel and for the resonance level (we go back to the original $d^\dagger, d$ operators), we need two equations of motion,

$$\begin{aligned} i\partial_t\psi_-(x) &= -i\partial_x\psi_-(x) + \delta(x)(\gamma_L d - \gamma_R d^\dagger), \\ i\partial_t d &= \Delta d + \gamma_L\psi_-(0) + \gamma_R\psi_-^\dagger(0). \end{aligned} \tag{6}$$

Integrating the first one around $x = 0$, we obtain $i[\psi_-(0^+) - \psi_-(0^-)] = \gamma_L d - \gamma_R d^\dagger$. Acting with $\partial_t^2 + \Delta^2$ on both sides of this relation yields

$$(\partial_t^2 + \Delta^2)[\psi_-(0^+) - \psi_-(0^-)] = -[(\gamma_L^2 + \gamma_R^2)\partial_t + i\Delta(\gamma_R^2 - \gamma_L^2)]\psi_-(0) - 2\gamma_R\gamma_L\partial_t\psi_-^\dagger(0). \tag{7}$$

Now we can insert into this relation the momentum decomposition of the field operator ψ_-:

$$\psi_-(x,t) = \int \frac{dk}{2\pi} e^{ik(t-x)} \begin{cases} a_k & \text{for } x < 0 \\ b_k & \text{for } x > 0. \end{cases} \tag{8}$$

Because the dispersion relation is linear, $\omega = vk = k$, we can use ω as the momentum variable as well as the energy variable. Inserting Eq. (8) into Eq. (7) and using $\psi_-(0) = [\psi_-(0^+) + \psi_-(0^-)]/2$ results in

$$E(b_\omega - a_\omega) = -i\beta_+(a_\omega + b_\omega) + i\gamma(a^\dagger_{-\omega} + b^\dagger_{-\omega}), \quad (9)$$

where we introduced the following objects: $E = \Delta^2 - \omega^2$, $\beta_\pm = [(1-2\alpha)\Delta \pm \omega]/2$, $\gamma = \omega\sqrt{\alpha(1-\alpha)}$, and $\alpha = \gamma_L^2/(\gamma_L^2 + \gamma_R^2)$ (the asymmetry parameter). From now on, ω, Δ, the bias voltage V, and the temperature T are all measured in units of $\Gamma = \gamma_L^2 + \gamma_R^2$. Considering in addition to Eq. (9) its complex conjugate for $-\omega$, we establish a relation between the amplitudes of the incoming and transmitted particle fluxes. The transmission coefficient can then be read off as follows:

$$T(\omega) = \frac{4\gamma^2 E^2}{(E^2 + \beta_+^2)(E^2 + \beta_-^2) + 2\gamma^2(E^2 + \beta_-\beta_+) + \gamma^4}, \quad (10)$$

This equation, accompanied by Eq. (5), provides all the information about the transport properties of the system and is the central result of this paper. The experimentally relevant quantity is the differential conductance $G = dI/dV$. At zero temperature, Eq. (5) considerably simplifies and one finds $G/G_0 = T(V)$. In the case when the couplings between the dot and the leads are perfectly symmetric and one of the chemical potentials matches Δ, G reaches the maximal value of G_0. This is a typical signature of the resonant tunneling effect usually encountered in transport phenomena in double-barrier structures [13].

The interplay between the LLs enhancement of the backscattering at low temperatures (resulting in decreasing conductance) and the more standard Breit-Wigner physics emerging in the resonant tunneling can be seen in Fig. 1. As predicted in Refs. [3,4], in the symmetric case $\alpha = 0.5$ and for $\Delta = 0$, the conductance saturates at low temperatures to its maximal value. In the presence of an asymmetry G does not saturate any more and vanishes as a power law towards $T = 0$ with the exponent 2. This value is equal to twice the density of states exponent ν of the LL with an open boundary: $\nu = 1/g - 1$ [8], which in our case is equal to 1. This fact indicates that in this regime the electrons are transferred through the system in a single stage process [4], so that the internal structure of the dot does not matter any more.

In accordance with the results of [3], the high temperature ($T \gg 1$) evolution of the conductance follows the law $G/G_0 \sim 1/T$. The reason is that the problem maps onto a free-fermion one, for which the $1/T$ behavior is inevitable. This result cannot be reproduced within the approach of NG, which applies to systems with weak interactions only.

Another interesting issue is the resonance width at half maximum, $w(T)$, as a function of temperature (see Fig. 2). At high temperatures, it decreases linearly upon lowering T no matter how strong the interactions. For $T \ll 1$, however, the correlation effects become visible and the width $w(T)$ of the peak saturates at $T = 0$ unless the dot is symmetric. In the latter case, $w(T)$ shrinks to zero with the exponent $1 - g$ predicted in Ref. [3]: $w(T) \sim T^{0.5}$.

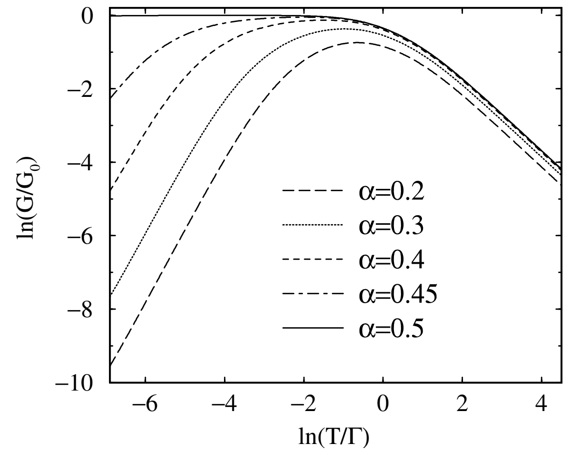

FIG. 1. Linear differential conductance at the resonance $\Delta = 0$ as a function of temperature for different values of the asymmetry parameter α.

It is not difficult to evaluate the integral in Eq. (5) analytically. However, we shall only present here some particular cases. To begin with, we observe that there is an intimate relation between our model at $\Delta = 0$, $\alpha = 1/2$ (resonant and symmetric case), and the $g = 1/2$ solution for the conductance through a *single* barrier (G_s) given in [3]. Indeed, evaluating (10) for the case in question, we find (in linear response)

$$G_{\Delta=0}(T)/G_0 = \frac{1}{2\pi T}\psi'\left(\frac{1}{2} + \frac{1}{2\pi T}\right), \quad (11)$$

where ψ is the ψ function. Comparing with Ref. [14] (see also [8]), we observe that $G_{\Delta=0}(T)/G_0 = 1 - G_s(T)/G_0$ if T in G_s is measured in units of the backscattering strength. One can easily show that an analogous relation continues to hold for the out-of-equilibrium current.

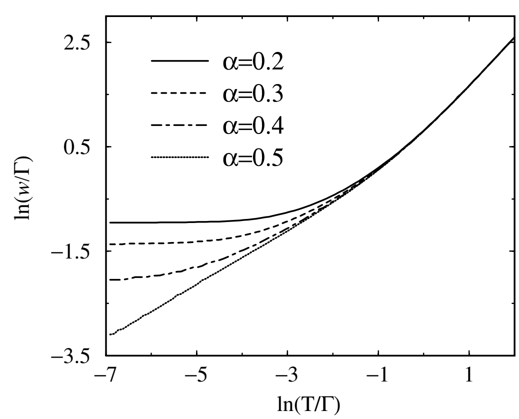

FIG. 2. The width of the resonant conductance peak ($\Delta = 0$) as a function of temperature for different values of the asymmetry parameter α.

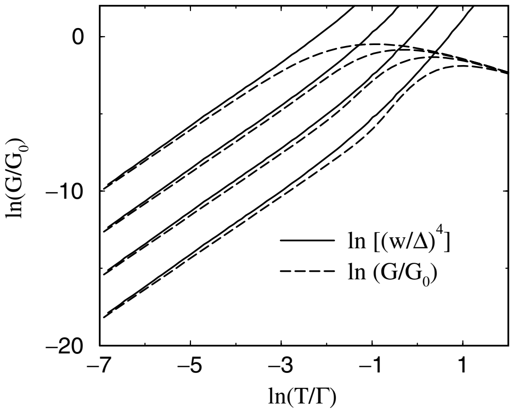

FIG. 3. Comparison between the linear conductance (dashed lines) and the approximative scaling function $(w/\Delta)^4$ (solid lines) for different values of Δ: from above $\Delta = 0.5$, 1, 2, 4.

Furthermore, for the linear conductance when $\alpha = 1/2$ but $\Delta \neq 0$ ($|\Delta| < 1/2$), we obtain

$$G_\Delta(T)/G_0 = \frac{1}{2\pi T(\lambda_+^2 - \lambda_-^2)} \times \left[\lambda_+ \psi'\left(\frac{1}{2} + \frac{\lambda_+}{2\pi T}\right) - \lambda_- \psi'\left(\frac{1}{2} + \frac{\lambda_-}{2\pi T}\right)\right], \tag{12}$$

where $\sqrt{2}\lambda_\pm = \sqrt{1 - 2\Delta^2 \pm \sqrt{1 - 4\Delta^2}}$. The same formula is valid in the resonant but asymmetric case ($\Delta = 0$, $\alpha \neq 1/2$) if we substitute $\lambda_\pm \to 1/2 \pm \sqrt{\alpha(1-\alpha)}$. This shows that the asymmetry parameter is equivalent to a small offset of the resonance (if the two act independently). (Indeed, in terms of the Kondo analogy, the channel asymmetry is known to be relevant.) For a strong offset of the resonance ($|\Delta| > 1/2$), formula (12) is not valid and should be replaced by

$$G_\Delta(T)/G_0 = \frac{\Delta}{\pi T\sqrt{4\Delta^2 - 1}} \mathrm{Im}\left[e^{i\theta}\psi'\left(\frac{1}{2} + \frac{\Delta e^{i\theta}}{2\pi T}\right)\right], \tag{13}$$

where $\theta = \tan^{-1}[\sqrt{4\Delta^2 - 1}/(1 - 2\Delta^2)]/2$.

As pointed out in Ref. [3], for $\Delta, T \ll 1$, the conductance should become a universal (scaling) function of the ratio of the resonance width and the backscattering strength (resonance offset). Indeed, taking the appropriate limit in formula (12), we obtain the exact scaling function at $g = 1/2$: $G_\Delta(T)/G_0 = \tilde{G}_{g=1/2}(X)$, where

$$\tilde{G}_{1/2}(X) = 1 - \frac{2}{\pi^2} X^2 \psi'\left(\frac{1}{2} + \frac{2}{\pi^2} X^2\right), \tag{14}$$

and the scaling variable is $X = \sqrt{\pi}\Delta/2T^{1/2}$ [that is, $w(T) \simeq 2T^{1/2}/\sqrt{\pi}$ at small T]. We note that our scaling function $\tilde{G}_{g=1/2}(X)$ is by far more complicated than the one found by NG in weak coupling. Furthermore, we observe from our analytic expressions that beyond $\Delta, T \ll 1$ there is *no* exact scaling. However, upon determining $w(T)$ numerically and plotting $[w(T)/\Delta]^4$ versus the dimensionless conductance G/G_0, we obtain an *approximate* numerical scaling as shown in Fig. 3. The same scaling function (14) holds in the resonant ($\Delta = 0$) but weakly asymmetric case, when $\alpha - 1/2$ is small. In that situation, $\alpha - 1/2$ substitutes Δ.

In summary, we presented an explicit solution for the transport through a resonant level coupled to two LL leads. It turns out that for $g = 1/2$ the Hamiltonian of the system can be mapped onto one similar to the two-channel Kondo Hamiltonian in the Toulouse limit, solvable exactly. We obtained the full *I-V* characteristics, which shows all the effects inherent to resonant tunneling setups in LLs, including the scaling. Our solution confirms previous results obtained by means of the perturbation theory and goes beyond them. In the future, it would be interesting to study deviations from $g = 1/2$ (in the spirit of Ref. [14]) and to investigate the effects of electron spin (and flavor).

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