Comment on "Linear Superposition in Nonlinear Equations"

In a recent Letter [1], Khare and Sukhatme (KS) have pointed out an apparently interesting and surprising property of the periodic Jacobi elliptic function solutions of the Korteweg–deVries (KdV) and other nonlinear evolution equations, namely, that for every such solution a specific type of superposition of shifted periodic solutions is also a solution. It is the aim of this Comment to clarify that this apparently new solution is essentially the original periodic solution in disguise with a changed velocity, and fundamentally no new solution results.

Following KS, let us consider the KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

and its simplest periodic solution

$$u_1(x,t) = -2\alpha^2 \mathrm{dn}^2(\xi_1, m) + \beta\alpha^2, \tag{2}$$

where $\xi_1 = \alpha(x - b_1\alpha^2 t)$, $b_1 = 8 - 4m - 6\beta$. According to KS, the following linear combination of shifted periodic solutions (2) yields a new periodic solution:

$$u_p(x,t) = -2\alpha^2 \sum_{i=1}^{p} d_i^2 + \beta\alpha^2, \tag{3}$$

where $d_i = \mathrm{dn}[\xi_p + 2(i-1)K(m)/p, m]$, $\xi_p = \alpha(x - b_p\alpha^2 t)$, $p = 2, 3, \ldots$.

To clarify this point, let us consider the multiperiodic (multiphase) solution of the KdV equation [2]

$$u(x,t) = -2\partial_x^2 \ln\Theta(z|B), \tag{4}$$

where $\Theta(z|B)$ denotes the Riemann theta function of genus g. For $g > 1$, B is a $g \times g$ Riemann matrix and $z = kx - \omega t$ is a vector. However, in the one-periodic case ($g = 1$), corresponding to that of KS, both z and B are numbers, and Eq. (4) becomes equivalent to Eq. (2). (The constant background $\beta\alpha^2$ can be removed by a suitable Galilean transformation; thus, without loss of generality we can put $\beta = 0$.) We note here that the theta function formalism has an additional practical advantage—the quasiperiodicity of the theta function [3] implies that the spatial (or temporal) period of the one-periodic solution is directly proportional to B, a fact that makes the discussion of the KS solution much easier.

The following identities (valid for $g = 1$) are crucial for further analysis:

$$\begin{aligned} &\Theta(z - B/4|B)\Theta(z + B/4|B) = C_2\Theta(z|B/2), \\ &\Theta(z - B/3|B)\Theta(z|B)\Theta(z + B/3|B) = C_3\Theta(z|B/3), \end{aligned} \tag{5}$$

and so on, where C_p is a theta constant of genus $(p-1)$, $p = 2, 3, \ldots$. The relations (5) can be easily verified for arbitrary p by using Jacobi's infinite product formula for the theta function [3]. On the other hand, Eqs. (5) are closely related to the generalized addition theorem reported recently [4].

The structure of Eqs. (5) is clear. The left-hand side represents a product of $p = 2, 3, \ldots$ theta functions of the same B, but shifted in argument z by $1/p$ of the period. The right-hand side is proportional to a single theta function of the same z, but of the period B/p. Substituting Eq. (5) into Eq. (4) we obtain the sum of p shifted periodic solutions, which is equal to the solution of the same argument z but of the period p times shorter. This is exactly the situation described by the KS solution [Eq. (3)]. In other words, Eq. (3) is not a new solution, but merely a different form of the well-known periodic solution of the KdV equation.

The same analysis can be repeated for other nonlinear "soliton" systems, such as mKdV and also ϕ^4. To apply the transformation (5) it is sufficient to know a periodic solution, expressed by the theta functions (or, equivalently, by Jacobi elliptic functions).

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