

Universal Scaling Behavior at the Upper Critical Dimension of Nonequilibrium Continuous Phase Transitions

S. Lübeck

Department of Physics of Complex Systems, Weizmann Institute of Science, 76100 Rehovot, Israel

P. C. Heger

Institut für Theoretische Physik, Universität Duisburg-Essen, 47048 Duisburg, Germany

(Received 12 December 2002; published 12 June 2003)

In this work we analyze the universal scaling functions and the critical exponents at the upper critical dimension of a continuous phase transition. The consideration of the universal scaling behavior yields a decisive check of the value of the upper critical dimension. We apply our method to a nonequilibrium continuous phase transition. By focusing on the equation of state of the phase transition it is easy to extend our analysis to all equilibrium and nonequilibrium phase transitions observed numerically or experimentally.

DOI: 10.1103/PhysRevLett.90.230601

PACS numbers: 05.70.Ln, 05.50.+q, 05.65.+b

One of the most impressive features of continuous phase transitions is the concept of universality that allows one to group the great variety of different types of critical phenomena into a small number of universality classes (see [1] for a recent review). All systems belonging to a given universality class have the same critical exponents and the corresponding scaling functions (equation of state, correlation functions, etc.) become identical near the critical point. Classical examples of such universal behavior are for instance the coexistence curve of liquid-vapor systems [2] and the equation of state in ferromagnetic systems (see for instance [1,3]). Checking the universality class it is often a more exacting test to consider scaling functions and amplitude combinations (which are just particular values of the scaling functions) rather than the values of the critical exponents. While for the latter ones the variations between different universal classes are often small, the amplitude combinations and therefore the scaling functions may differ significantly (see [4]). A foundation for the understanding of the concept of universality as well as a tool to estimate the values of the critical exponents was provided by Wilson's renormalization group (RG) approach [5,6] which maps the critical point onto a fixed point of a certain transformation of the system's Hamiltonian, Langevin equation, etc.

Furthermore the RG explains the existence of an upper critical dimension D_c above which the mean-field theory applies, whereas it fails below D_c . At the upper critical dimension the RG equations yield mean-field exponents with logarithmic corrections [7]. These logarithmic corrections make the data analysis quite difficult and thus most investigations are focused on the determination of the correction exponents [see Eqs. (5) and (6) below] only, lacking the determination of the scaling functions.

In this work we investigate the universal scaling behavior of a continuous phase transition at D_c and develop a method of analysis that allows us to determine the exponents as well as the scaling functions. Therefore we

consider three different nonequilibrium systems exhibiting a continuous phase transition into an absorbing phase. Focusing on the equation of state our method can be easily applied to all equilibrium as well as nonequilibrium continuous phase transitions observed in numerical simulations or experiments (as long as the conjugated field can be physically realized). In all three models the dynamics obey particle conservation and, according to the universality hypothesis of [8], all models are expected to belong to the universality class of absorbing phase transitions with a conserved field.

The first considered model is the conserved lattice gas (CLG) which was introduced in [8]. In the CLG lattice sites may be empty or occupied by one particle. In order to mimic a repulsive interaction a given particle is considered as active if at least one of its neighboring sites on the lattice is occupied by another particle. If all neighboring sites are empty the particle remains inactive. Active particles are moved in the next update step to one of their empty nearest neighbor sites, selected at random.

The second model is the so-called conserved transfer threshold process (CTTP) [8]. Here, lattice sites may be empty, occupied by one particle, or occupied by two particles. Empty and single occupied sites are considered as inactive whereas double occupied lattice sites are considered as active. In the latter case one tries to transfer both particles of a given active site to randomly chosen empty or single occupied nearest neighbor sites.

The third model is a modified version of the Manna sandpile model [9], the so-called fixed-energy Manna model [10]. In contrast to the CTTP the Manna model allows unlimited particle occupation of lattice sites. All lattice sites which are occupied by at least two particles are considered as active and all particles are moved to the neighboring sites selected at random.

In our simulations (see [11,12] for details) we start from a random distribution of particles and all models reach after a transient regime a steady state which is

characterized by the density of active sites ρ_a . The density ρ_a is the order parameter and the particle density ρ is the control parameter of the absorbing phase transition, i.e., the order parameter vanishes at the critical density ρ_c according to $\rho_a \propto \delta\rho^\beta$, with the reduced control parameter $\delta\rho = \rho/\rho_c - 1$. Additionally to the order parameter we consider its fluctuations $\Delta\rho_a$. Approaching the transition point from above ($\delta\rho > 0$) the fluctuations diverge according to $\Delta\rho_a \propto \delta\rho^{-\gamma'}$ (see [11,12]). Below the critical density (in the absorbing state) the order parameter as well as its fluctuations are zero in the steady state.

Similar to equilibrium phase transitions it is possible in the case of absorbing phase transitions to apply an external field h which is conjugated to the order parameter, i.e., the field causes a spontaneous creation of active particles (see for instance [13]). A realization of the external field for absorbing phase transitions with a conserved field was recently developed in [11] where the external field triggers movements of inactive particles which may be activated in this way. At the critical density ρ_c the order parameter and its fluctuations scale as $\rho_a \propto h^{\beta/\sigma}$ and $\Delta\rho_a \propto h^{-\gamma'/\sigma}$, respectively.

Before we focus our attention to the scaling behavior at the upper critical dimension D_c we briefly reconsider the scaling behavior below and above D_c . In both cases the order parameter obeys for all positive values of λ the universal scaling ansatz

$$a_a \rho_a(\delta\rho, h) \sim \lambda^{-\beta} \tilde{R}(a_\rho \delta\rho \lambda, a_h h \lambda^\sigma). \quad (1)$$

The universal scaling function $\tilde{R}(x, y)$ is the same for all systems belonging to a given universality class whereas all nonuniversal system-dependent features (e.g., the lattice structure, the range of interaction, the update scheme, etc.) are contained in the so-called nonuniversal metric factors a_a , a_ρ , and a_h [14]. Using the transformation $\lambda \rightarrow a_a^{-1/\beta} \lambda$, the number of metric factors can be reduced to $c_\rho = a_\rho a_a^{-1/\beta}$ and $c_h = a_h a_a^{-\sigma/\beta}$. We will see that this simple reduction is not possible at the upper critical dimension D_c . Thus instead of this transformation we set in the following $a_a = 1$ for $D \neq D_c$ in order to formulate for all dimensions a unified universal scaling scheme.

The universal scaling function \tilde{R} is normed by the conditions $\tilde{R}(1, 0) = \tilde{R}(0, 1) = 1$ and the nonuniversal metric factors can be determined from the amplitudes of $\rho_a(\delta\rho, h = 0) \sim (a_\rho \delta\rho)^\beta$ and $\rho_a(\delta\rho = 0, h) \sim (a_h h)^{\beta/\sigma}$. These equations are obtained by choosing in the scaling ansatz Eq. (1) $a_\rho \delta\rho \lambda = 1$ and $a_h h \lambda^\sigma = 1$, respectively. Furthermore, the choice $a_h h \lambda^\sigma = 1$ leads to the well-known scaling equation of the order parameter

$$\rho_a(\delta\rho, h) \sim (a_h h)^{\beta/\sigma} \tilde{R}[a_\rho \delta\rho (a_h h)^{-1/\sigma}, 1]. \quad (2)$$

Thus plotting the rescaled order parameter $(a_h h)^{-\beta/\sigma} \rho_a$ as a function of the rescaled control parameter $a_\rho \delta\rho (a_h h)^{-1/\sigma}$ the corresponding data of all systems in a given universality class have to collapse onto the single

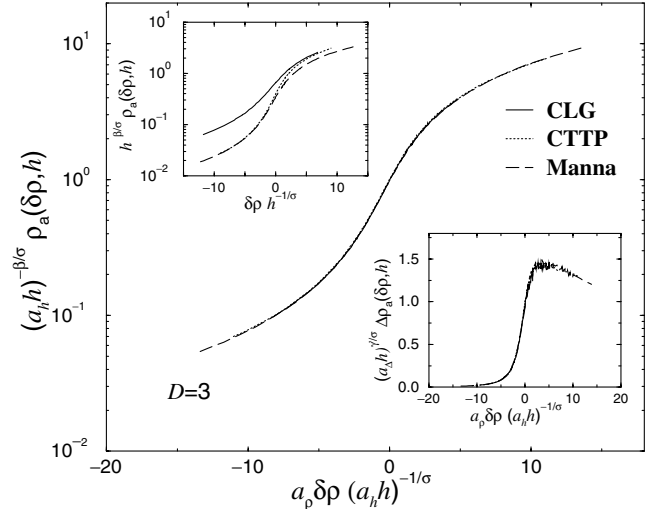


FIG. 1. The universal scaling function of the order parameter and its fluctuations (lower right inset) for the CLG model, the CTP, as well as the Manna model for $D = 3$ with $\beta = 0.840$ and $\sigma = 2.069$. The values of the nonuniversal metric factors are listed in Table I. The upper left inset displays the nonuniversal scaling plots accordingly neglecting the nonuniversal metric factors. For all considered models the scaling plots contain at least four different curves corresponding to four different field values (see [11,12] for details).

curve $\tilde{R}(x, 1)$. This is shown in Fig. 1 for the CLG model, the CTP, and the Manna model for $D = 3$. In the case that metric factors are neglected one observes the nonuniversal scaling behavior where each model is characterized by its own scaling function (see the left inset of Fig. 1).

Similarly the order parameter fluctuations are expected to obey the scaling ansatz

$$a_\Delta \Delta\rho_a(\delta\rho, h) = \lambda^{\gamma'} \tilde{D}(a_\rho \delta\rho \lambda, a_h h \lambda^\sigma). \quad (3)$$

Again the number of metric factors can be reduced by a simple transformation to $d_\rho = a_\rho a_\Delta^{1/\gamma'}$ and $d_h = a_h a_\Delta^{\sigma/\gamma'}$. But it is instructive to use the above ansatz [Eq. (3)] since exactly one new metric factor (a_Δ) is introduced for the fluctuations, and furthermore the universal functions \tilde{R} and \tilde{D} are characterized by the same metric factors. Identical metric factors for \tilde{R} and \tilde{D} occur for instance naturally in equilibrium thermodynamics where both functions can be in principle derived from a single thermodynamic potential, e.g., the free energy. In the case of nonequilibrium phase transitions one can argue that both functions can be derived from a corresponding Langevin equation. Setting $\tilde{D}(0, 1) = 1$ the nonuniversal metric factor a_Δ can be determined by the amplitude of the divergence of $\Delta\rho_a$ similar to the order parameter. In the right inset of Fig. 1 we plot the rescaled fluctuations as a function of the rescaled order parameter, i.e., the universal scaling function $\tilde{D}(x, 1)$. Similar to the equation of state we get a good data collapse of the corresponding data.

TABLE I. The nonuniversal quantities for various dimensions. The uncertainty of the metric factors is less than 5%. For greater uncertainties the corresponding data sets display significant deviations from the presented universal scaling plots.

Model	D	ρ_c	a_a	a_ρ	a_h	a_Δ
CLG	3	$0.217\,91 \pm 0.000\,09$	1	0.434	0.391	8.881
CTTP	3	$0.604\,89 \pm 0.000\,02$	1	0.384	0.093	24.51
Manna	3	$0.600\,18 \pm 0.000\,04$	1	0.311	0.074	32.24
CLG	4	$0.157\,05 \pm 0.000\,10$	4.307	1.664	8.021	7.327
CTTP	4	$0.567\,05 \pm 0.000\,03$	0.689	0.269	0.047	17.18
Manna	4	$0.564\,51 \pm 0.000\,07$	0.690	0.245	0.040	18.82
CLG	5	$0.122\,98 \pm 0.000\,15$	1	0.329	0.665	8.971
CTTP	5	$0.548\,64 \pm 0.000\,05$	1	0.461	0.251	18.73
CTTP	6	$0.538\,16 \pm 0.000\,07$	1	0.421	0.218	157.5
Manna	5	$0.547\,04 \pm 0.000\,09$	1	0.870	0.225	20.69

We consider now the scaling behavior above the upper critical dimension D_c . According to the renormalization group scenario the stable fixed point of the renormalization equations is usually the trivial fixed point with classical (mean-field) universal quantities. Thus, in contrast to the situation below D_c the critical exponents as well as the universal scaling functions are independent of the particular value of the dimension for $D > D_c$. In most cases it is possible to derive these mean-field exponents and even the scaling functions exactly since correlations and fluctuations can be neglected above D_c . The mean-field scaling behavior of the CLG model and the CTTP was considered in [15] and agrees with that of directed percolation, i.e., the scaling functions are given by [15,16] $\tilde{R}(x, y) = x/2 + [y + (x/2)^2]^{1/2}$ and $\tilde{D}(x, y) = \tilde{R}(x, y)[y + (x/2)^2]^{-1/2}$. One can easily show that $\beta = 1$, $\sigma = 2$, and $\gamma' = 0$. The latter case corresponds to a jump of the fluctuations at the critical point which was already observed in numerical simulations [11,12].

In Fig. 2 we plot the rescaled order parameter as well as the rescaled order parameter fluctuations for $D = 5$ and $D = 6$. In all cases the numerical data are in perfect agreement with the mean-field scaling functions $\tilde{R}(x, 1)$ and $\tilde{D}(x, 1)$, respectively. Thus we clearly get the upper bound for the critical dimension, namely, $D_c < 5$. This is a nontrivial result since a recently performed phenomenological field theory predicts the too large value $D_c = 6$ [17].

We now address the question of the scaling behavior at the upper critical dimension $D_c = 4$. Here the scaling behavior is governed by the mean-field exponents modified by logarithmic corrections. For instance the order parameter obeys in leading order $\rho_a(\delta\rho, h = 0) \propto \delta\rho |\ln\delta\rho|^B$ and $\rho_a(\delta\rho = 0, h) \propto \sqrt{h} |\ln h|^\Sigma$, respectively. The logarithmic correction exponents B and Σ are characteristic features of the whole universality class similar to the usual critical exponents. Thus it was rather surprising that recent numerical investigations of the CLG model ($B = 0.24$, $\Sigma = 0.45$) and of the CTTP ($B = 0.15$, $\Sigma = 0.28$) reveals different values of the logarithmic correction exponents [12]. In the following we will develop a complete scaling scenario at the upper critical

dimension which agrees which the RG conjecture, i.e., all considered models are characterized by the same critical exponents, the same logarithmic correction exponents, as well as the same universal scaling functions.

As argued in [11] we assume that the universal scaling ansatz of the order parameter obeys in leading order

$$a_a \rho_a(\delta\rho, h) \sim \lambda^{-\beta} |\ln\lambda|^l \tilde{R}(a_\rho \delta\rho \lambda |\ln\lambda|^b, a_h h \lambda^\sigma |\ln\lambda|^s). \quad (4)$$

Thus the order parameter at zero field ($h = 0$) and at the critical density ($\delta\rho = 0$) is given in leading order by

$$a_a \rho_a(\delta\rho, h = 0) \sim a_\rho \delta\rho |\ln a_\rho \delta\rho|^B \tilde{R}(1, 0), \quad (5)$$

$$a_a \rho_a(\delta\rho = 0, h) \sim \sqrt{a_h h} |\ln \sqrt{a_h h}|^\Sigma \tilde{R}(0, 1), \quad (6)$$

with $B = b + l$ and $\Sigma = s/2 + l$ and where we use the mean-field values $\beta = 1$ and $\sigma = 2$, respectively. Similar to the case $D \neq D_c$ we set again $\tilde{R}(0, 1) = \tilde{R}(1, 0) = 1$.

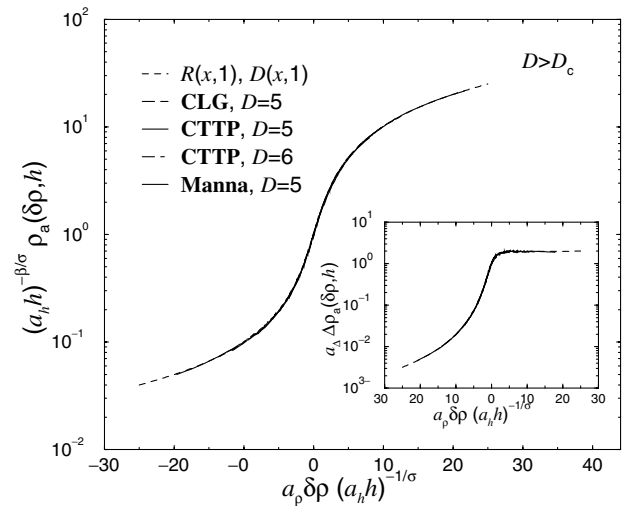


FIG. 2. The universal scaling function of the order parameter and its fluctuations (inset) above the upper critical dimension $D_c = 4$ with $\beta = 1$ and $\sigma = 2$. The numerical data agree perfectly with the universal mean-field scaling functions $\tilde{R}(x, 1)$ and $\tilde{D}(x, 1)$ (thick dashed lines).

Although the universal scaling ansatz [Eqs. (4)–(6)] and the nonuniversal scaling ansatz (without metric factors) are asymptotically equal, they may lead to different results for numerically available data. For instance the nonuniversal metric factor in Eq. (5) results in the correction factor $|1 + \ln a_\rho / \ln \delta\rho|^B$ compared to the nonuniversal ansatz. This factor tends to 1 for $\delta\rho \rightarrow 0$ but in numerical simulations $\delta\rho$ is hardly smaller than 10^{-3} , which explains why different values of B and Σ are observed numerically [12].

According to the ansatz Eq. (4) the scaling behavior of the equation of state is given in leading order by

$$a_a \rho_a(\delta\rho, h) \sim \sqrt{a_h h} |\ln \sqrt{a_h h}|^\Sigma \tilde{R}(x, 1), \quad (7)$$

where the scaling argument is given in leading order by $x = a_\rho \delta\rho \sqrt{a_h h}^{-1} |\ln \sqrt{a_h h}|^\Xi$ with $\Xi = b - s/2 = B - \Sigma$. Similarly we use for the order parameter fluctuations the ansatz

$$a_\Delta \Delta \rho_a(\delta\rho, h) \sim \lambda^{\gamma'} |\ln \lambda|^k \tilde{D}(a_\rho \delta\rho \lambda |\ln \lambda|^b, a_h h \lambda^{-\sigma} |\ln \lambda|^s). \quad (8)$$

Using the mean-field value $\gamma' = 0$ and taking into account that the order parameter fluctuations remain finite at D_c [11,12] (i.e., $k = 0$) we get the scaling function $a_\Delta \Delta \rho_a(\delta\rho, h) \sim \tilde{D}(x, 1)$. The nonuniversal metric factor a_Δ is determined by the condition $\tilde{D}(0, 1) = 1$.

Thus the scaling behavior of the order parameter and its fluctuations at the upper critical dimension is determined by two independent exponents (B and Σ) and four nonuniversal metric factors ($a_a, a_\rho, a_h, a_\Delta$). We determine these values in our analysis by the following conditions which are applied simultaneously: first, both the rescaled equation of state and the rescaled order parameter fluctuations have to collapse to the universal functions $\tilde{R}(x, 1)$ and $\tilde{D}(x, 1)$ for all considered models. Second, the order parameter behavior at zero field and at the critical density is asymptotically given by the simple function $f(x) = x$ if one plots $[a_a \rho_a(\delta\rho, 0) / a_\rho \delta\rho]^{1/B}$ vs $|\ln a_\rho \delta\rho|$ and $[a_a \rho_a(0, h) / \sqrt{a_h h}]^{1/\Sigma}$ vs $|\ln \sqrt{a_h h}|$, respectively. Applying this analysis we observed that convincing results are obtained for $\Sigma = 0.35$ and $B = 0.20$ (see Table I for the values of the nonuniversal scaling factors). The corresponding plots are presented in Fig. 3. In particular the data collapse of the equation of state is quite sensitive for variations of the exponents B and Σ . Thus the quality of the corresponding data collapse could be used in order to estimate the error bars of the logarithmic correction exponents. We obtained in this way $\Sigma = 0.35 \pm 0.06$ and $B = 0.20 \pm 0.05$.

In conclusion, the investigation of the universal scaling behavior presents reliable results of the logarithmic correction exponents in contrast to the nonuniversal scaling analysis. Furthermore the universal scaling analysis allows one to determine the value of D_c just by checking

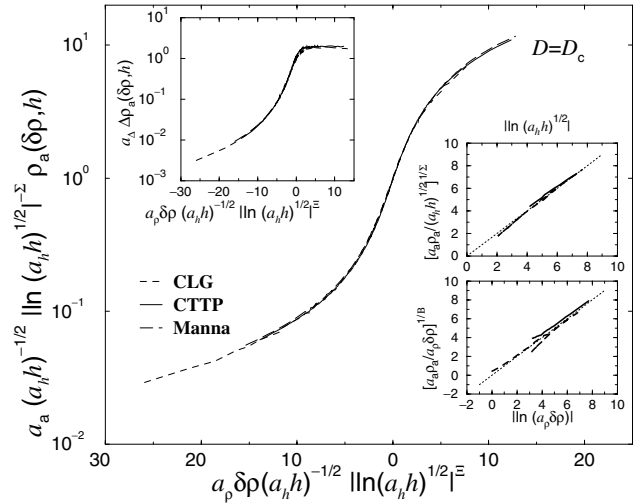


FIG. 3. The universal scaling function at the upper critical dimension $D_c = 4$. The right insets show the order parameter at the critical density and for zero field, respectively. The order parameter is rescaled according to Eqs. (5) and (6). Approaching the transition point ($h \rightarrow 0$ and $\delta\rho \rightarrow 0$) the data tend to the function $f(x) = x$ (dotted lines) as required.

whether the numerical or experimental data are in agreement with the usually known universal mean-field scaling functions.

We would like to thank A. Hucht and P. K. Mohanty for helpful discussions. This work was financially supported by the Minerva Foundation (Max Planck Gesellschaft).

-
- [1] H. E. Stanley, *Rev. Mod. Phys.* **71**, S358 (1999).
 - [2] E. A. Guggenheim, *J. Chem. Phys.* **13**, 253 (1945).
 - [3] S. Milošević and H. E. Stanley, *Phys. Rev. B* **6**, 1002 (1972).
 - [4] V. Privman, P. C. Hohenberg, and A. Aharony, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 1991), Vol. 14.
 - [5] K. G. Wilson, *Phys. Rev. B* **4**, 3174 (1971).
 - [6] K. G. Wilson, *Phys. Rev. B* **4**, 3184 (1971).
 - [7] K. G. Wilson and J. Kogut, *Phys. Rep.* **12C**, 75 (1974).
 - [8] M. Rossi, R. Pastor-Satorras, and A. Vespignani, *Phys. Rev. Lett.* **85**, 1803 (2000).
 - [9] S. S. Manna, *J. Phys. A* **24**, L363 (1991).
 - [10] A. Vespignani, R. Dickman, M. A. Muñoz, and S. Zapperi, *Phys. Rev. E* **62**, 4564 (2000).
 - [11] S. Lübeck, *Phys. Rev. E* **65**, 046150 (2002).
 - [12] S. Lübeck, *Phys. Rev. E* **66**, 046114 (2002).
 - [13] H. Hinrichsen, *Adv. Phys.* **49**, 815 (2000).
 - [14] V. Privman and M. E. Fisher, *Phys. Rev. B* **30**, 322 (1984).
 - [15] S. Lübeck and A. Hucht, *J. Phys. A* **35**, 4853 (2002).
 - [16] H. Mori and K. J. McNeil, *Prog. Theor. Phys.* **57**, 770 (1977).
 - [17] F. van Wijland, *Phys. Rev. Lett.* **89**, 190602 (2002).