Adiabatic Dynamics of Periodic Waves in Bose-Einstein Condensates with Time Dependent Atomic Scattering Length

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Evolution of periodic matter waves in one-dimensional Bose-Einstein condensates with time-dependent scattering length is described. It is shown that variation of the effective nonlinearity is a powerful tool for controlled generation of bright and dark solitons starting with periodic waves.

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Observation of Bose-Einstein condensate (BEC) in gases of weakly interacting alkali metals have stimulated intensive studies of the nonlinear matter waves. A new area of physics—nonlinear matter waves and nonlinear atomic optics—was originated. Generation as well as dynamics of solitary wave pulses in BEC's is one of the most important related problems. Experimental observations of dark [1,2] and bright [3,4] solitons have recently been reported. Theoretically, several methods of creating solitary waves have been proposed. First of all, it is a modulational instability [5], which is a universal phenomenon of the nonlinear physics (especially intensively explored in nonlinear optics; see [6]). This method, however, cannot predict exactly the parameters of generated solitons. Another method, which is controllable in the above sense, is the so-called phase engineering [2], which consists of imposing an initial phase on a BEC and is appropriate for generating dark solitons. The phase imprinting, however, affects the whole background condensate which acquires nonzero initial velocity and starts to oscillate in a trap potential. The problem becomes even more complicated when one is interested in generating trains (or lattices) of solitons in BEC's.

In this Letter we show that a powerful tool for generating and managing matter soliton trains can be provided by variation of the effective nonlinearity, which in practical terms can be achieved by means of variation of the s-wave scattering length a_s due to the Feshbach resonance [7]:

$$a_s(t) = a\{1 + \Delta/[B_0 - B(t)]\}. \tag{1}$$

Here a is the asymptotic value of the scattering length far from resonance, $B(t)$ is the time-dependent external magnetic field, Δ is the width of the resonance, and B_0 is the resonant value of the magnetic field. Feshbach resonances have been observed in Na at 853 and 907 G [8], in ^{7}Li at 725 G [4], and in ^{85}Rb at 164 G with $\Delta = 11$ G [9]. Also, rapid variation in time of a_s has been recently used for generation of bright solitons in BEC [3,4]. Here we want to indicate that in quasi-one-dimensional geometry an initially weak modulation of the condensate can be amplified by means of proper variation of the scattering length. As a result, the condensate evolves into either a sequence of bright solitons for $a_s < 0$ or "domains" separated by dark solitons for $a_s > 0$. In the case of bright solitons the attractive forces between atoms exactly compensate the wave-packet dispersion in the longitudinal direction, so that the confining trap potential in this direction becomes unnecessary. Then the motion of bright solitons in the longitudinal direction can be controlled by means of application of external forces. Actually, oscillations of bright solitons in the trap observed in [3,4] give a simple example of such controllable motion. Thus, quasi-one-dimensional bright BEC solitons behave as separate entities, and their investigation seems to be a quite promising field of research. Dark solitons may be considered as moving "domain walls" which separate regions of a condensate with different values of the order parameter. Investigation of dark solitons is also useful for understanding the properties of BEC. In general, the problem of the controllable soliton generation is important for a number of BEC applications, like atomic interferometry [10], and different kinds of the quantum phase transitions [11], as well as in the context of the nonlinear physics, including nonlinear optics and hydrodynamics.

Our approach is based on the well established concept that the BEC dynamics at low enough temperature is well described by the three-dimensional (3D) Gross-Pitaevskii (GP) equation. In some physically important cases it admits a self-consistent reduction to the 1D nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} - \tfrac{1}{2}\nu^2 x^2 u - 2\sigma g|u|^2 u = 0. \tag{2}$$

In particular, this is the case of a cigar-shaped BEC of low density when $\epsilon = \mathcal{N}|a|/a_\perp \ll 1$ and $a_\perp^2/a_0^2 = \epsilon^2\nu/\sqrt{2}$, where $\mathcal{N}$ is a total number of atoms, $a_\perp = (\hbar/m_a\omega_\perp)^{1/2}$ and $a_0 = (\hbar/m_a\omega_0)^{1/2}$ are linear oscillator lengths in the transverse and in cigar-axis directions, respectively (in the small amplitude limit they are of order of effective sizes of the condensate), $\omega_\perp$ and ω_0 being respective harmonic oscillator frequencies, $\nu \lesssim 1$ is a positive parameter, and m_a is the atomic mass. In (2) time t and coordinate x are measured in units $2/(\epsilon^2\omega_\perp)$ and $a_\perp/\epsilon$, respectively. The order parameter in the leading order is searched in the form

$$\psi(\mathbf{r}, t) = \frac{\epsilon}{\sqrt{2\pi|a|}a_\perp}\exp\left(-i\omega_\perp t - \frac{y^2+z^2}{2a_\perp^2}\right)u\left(\frac{\epsilon x}{a_\perp}, \frac{\epsilon^2\omega_\perp t}{2}\right),$$

where $\sigma = \mathrm{sgn}(a_s)$, and $g(t) \equiv a_s(t)/a_s(0)$. It will be assumed that $a_s(t)$ does not change its sign and thus $g(t)$ is a positive-valued function. We notice that the smallness of the density rules out a possibility of collapse phenomenon (if $a < 0$).

We start with analytical estimates supposing that the initial wave function $u(x, 0)$ is modulated along the cigar axis with the wavelength L of modulation much less than the longitudinal dimension of the condensate, i.e., of the l: $L \ll l$. Therefore, at this stage we neglect the smooth trap potential and impose cyclic boundary conditions. Then the initial wave function can be approximated well enough by exact periodic solutions of Eq. (2) at $\nu = 0$. For example, if at $t = 0$ we take into account only one space harmonic of the initial wave function, $u(x, 0) = u_0 + u_1\cos(x/L)$, then this distribution can be approximated by well-known elliptic function solutions of Eq. (2) with a small parameter m (see below). We are interested in evolution of such solutions due to slow change of $g(t)$ with time. At the same time we suppose that the total time of adiabatically slow change of $g(t)$ is much less than the period $\sim 2\pi/\nu$ of soliton oscillations in the trap potential, so that we can neglect the influence of the trap potential on the motion of solitons during the formation of soliton trains and put $\nu = 0$ in Eq. (2). This means that we shall consider analytically relatively small segments of the modulated wave much greater than the wavelength L and much smaller than the size l of the whole condensate in the trap. To solve the problem of the condensate evolution, we note that substitution

$$u(x, t) = v(x, t)/\sqrt{g(t)} \tag{3}$$

transforms Eq. (2) with $\nu = 0$ into

$$iv_t + v_{xx} - 2\sigma|v|^2 v = i\varepsilon v \tag{4}$$

with $\varepsilon(t) = g'(t)/2g(t)$. Thus, for slowly varying $g(t)$ the right-hand side of Eq. (4) can be considered as a small perturbation: $|\varepsilon(t)| \ll 1$. As it follows from Eq. (3), for the initial distribution one has $v(x, 0) = u(x, 0)$. For our purposes it is enough to consider typical particular solutions of the unperturbed NLS equation which are parametrized by two parameters $\lambda_{1,2}$. Under influence of the perturbation, these parameters in the adiabatic approximation become slow functions of time t. Equations which govern their evolution can be derived by the following simple method.

First, the initial values of $\lambda_{1,2}$, as well as the coefficients in Eq. (4), are supposed to be independent of x; hence the wavelength L of the nonlinear wave evolving according to Eq. (4) is constant,

$$dL(\lambda_1(t), \lambda_2(t))/dt = 0. \tag{5}$$

Second, we find that the variable $N(\lambda_1(t), \lambda_2(t)) = \int_0^L |v|^2 dx$ changes with time according to

$$dN(\lambda_1(t), \lambda_2(t))/dt = 2\varepsilon N(\lambda_1(t), \lambda_2(t)). \tag{6}$$

Then, if the expressions for L and N in terms of $\lambda_{1,2}$ are known, Eqs. (5) and (6) reduce to two equations linear with respect to derivatives $d\lambda_{1,2}/dt$, which yield the desired system of differential equations for $\lambda_{1,2}$. The form of this system depends, of course, on the choice of the parameters $\lambda_{1,2}$. It is well known from the theory of modulations of nonlinear periodic waves that for completely integrable equations (as the NLS equation) the most convenient choice is provided by the so-called "finite-gap integration method" of obtaining periodic solutions. Therefore we shall use the parametrization of the periodic solutions of the NLS equation obtained by this method (see, e.g., [12]), and consider three most typical cases.

Case 1: cn-*wave in a BEC with a negative scattering length.*—In the case of a BEC with negative scattering length, $\sigma = -1$, there are two simple two-parametric periodic solutions of unperturbed Eq. (4). One of them has the form

$$v(x, t) = 2\lambda_1 e^{-4i(\lambda_1^2-\lambda_2^2)t}\mathrm{cn}[2\sqrt{\lambda_1^2+\lambda_2^2}\,x, m], \tag{7}$$

where the parameter of elliptic function is given by $m = \lambda_1^2/(\lambda_1^2+\lambda_2^2)$. Then straightforward calculations give

$$L = \frac{2K(m)}{\sqrt{\lambda_1^2+\lambda_2^2}}, \qquad N = 8\sqrt{\lambda_1^2+\lambda_2^2}\,E(m) - 4\lambda_2^2 L, \tag{8}$$

where $K(m)$ and $E(m)$ are complete elliptic integrals of the first and the second kinds, respectively. Substitution of these expressions into Eqs. (5) and (6) yields the system

$$\begin{aligned}\frac{d\lambda_1}{dz} &= \frac{[(\lambda_1^2+\lambda_2^2)E(m) - \lambda_2^2K(m)]E(m)\lambda_1}{\lambda_1^2E^2(m) + \lambda_2^2[K(m)-E(m)]^2},\\ \frac{d\lambda_2}{dz} &= \frac{[\lambda_2^2K(m) - (\lambda_1^2+\lambda_2^2)E(m)][K(m)-E(m)]\lambda_2}{\lambda_1^2E^2(m) + \lambda_2^2[K(m)-E(m)]^2},\end{aligned} \tag{9}$$

where

$$z = z(t) = 2\int_0^t \varepsilon(t')dt' = \ln g(t), \qquad z(0) = 0. \tag{10}$$

If dependence of λ_1 and λ_2 on z is found from (9), then Eq. (3) gives evolution of the periodic wave $u(x, t)$ with slow change of the parameter z connected with time t by Eq. (10). In particular, the density of particles in the condensate is given by

$$|u|^2 = 4e^{-z}\lambda_1^2(z)\,\mathrm{cn}^2[2\sqrt{\lambda_1^2(z) + \lambda_2^2(z)}\,x, m], \qquad (11)$$

where transformation to the time variable should be performed with the use of Eq. (10).

In Fig. 1(a) we present an example of the evolution of the respective density distribution in the presence of a harmonic trap potential where the parabolic parameter and experimentally feasible parameters are used. The figure shows that in the case of a negative scattering length given by (1) an increase of the magnetic field $B(t)$ within the region $B(0) < B(t) < B_0$ results in compression of the atomic density and formation of a lattice of matter solitons.

Case 2: dn-*wave in a BEC with a negative scattering length.*—Another simple solution of the NLS Eq. (4) with $\sigma = -1$ is given by

$$v(x, t) = (\lambda_1 + \lambda_2)e^{-2i(\lambda_1^2+\lambda_2^2)t}\mathrm{dn}[(\lambda_1 + \lambda_2)x, m], \qquad (12)$$

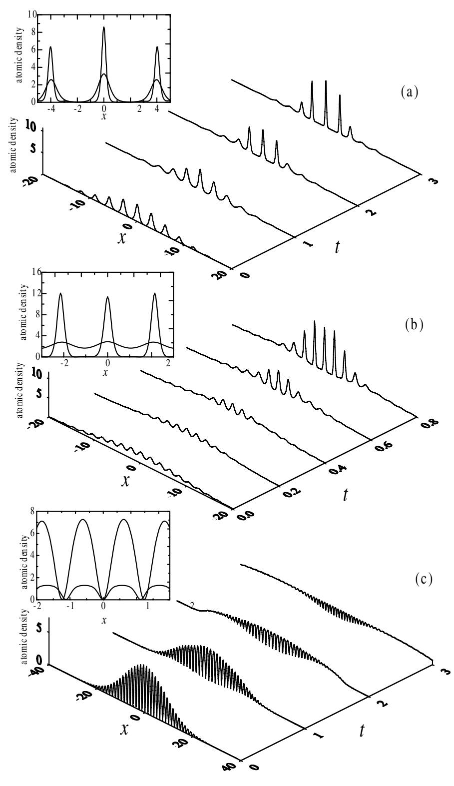

FIG. 1. Numerical solution of Eq. (2) with $g(t) = e^{t/\tau}$. Initial conditions are chosen in the form $u(x, 0)e^{-\nu x^2/2^{3/2}}$ where $u(x, 0)$ is given by (a) Eq. (7) with $\lambda_1(0) = 1$, $\lambda_2(0) = 0.2$, $\tau \approx 2$, and $\nu = 0.02$; (b) Eq. (12) with $\lambda_1(0) = 1.5$, $\lambda_2(0) = 0.2$, $\tau \approx 1.5$, and $\nu = 0.01$; (c) Eq. (15) with $\lambda_1(0) = 3$, $\lambda_2(0) = 0.3$, $\tau \approx 1$, and $\nu = 0.01$. In the insets we show density distributions at initial (thin lines) and final (thick lines) moments of time.

where $m = 4\lambda_1\lambda_2/(\lambda_1 + \lambda_2)^2$. By analogy with (9) we derive the following equations for λ_1 and λ_2:

$$\frac{d\lambda_1}{dz} = \frac{\lambda_1(\lambda_1 + \lambda_2)E(m)}{(\lambda_1 - \lambda_2)K(m) + (\lambda_1 + \lambda_2)E(m)},$$
$$\frac{d\lambda_2}{dz} = -\frac{\lambda_2(\lambda_1 + \lambda_2)E(m)}{(\lambda_1 - \lambda_2)K(m) - (\lambda_1 + \lambda_2)E(m)}, \qquad (13)$$

where it is supposed that $\lambda_1 > \lambda_2$ and z is defined by Eq. (10). Now the density of particles is given by

$$|u|^2 = e^{-z}(\lambda_1 + \lambda_2)^2\mathrm{dn}^2[(\lambda_1 + \lambda_2)x, m]. \qquad (14)$$

An example of the respective evolution in the presence of the potential is given in Fig. 1(b). One again observes formation of a lattice of matter solitons starting with a weakly modulated periodic wave.

Case 3: sn-*wave in a BEC with a positive scattering length.*—In the case of (4) with $\sigma = 1$ there exists simple periodic solution

$$v(x, t) = (\lambda_1 - \lambda_2)e^{2i(\lambda_1^2+\lambda_2^2)t}\mathrm{sn}((\lambda_1 + \lambda_2)x, m), \qquad (15)$$

where $m = [(\lambda_1 - \lambda_2)/(\lambda_1 + \lambda_2)]^2$ and it is supposed that $\lambda_1 > \lambda_2$. Now we obtain

$$\frac{d\lambda_1}{dz} = \frac{\lambda_1(\lambda_1 + \lambda_2)[K(m) - E(m)]}{2\lambda_1 K(m) - (\lambda_1 + \lambda_2)E(m)},$$
$$\frac{d\lambda_2}{dz} = \frac{\lambda_2(\lambda_1 + \lambda_2)[K(m) - E(m)]}{2\lambda_2 K(m) - (\lambda_1 + \lambda_2)E(m)}, \qquad (16)$$

with z defined again by Eq. (10). The density of particles in the condensate is given by

$$|u|^2 = e^{-z}(\lambda_1 - \lambda_2)^2\mathrm{sn}^2[(\lambda_1 + \lambda_2)x, m]. \qquad (17)$$

This case, but in the presence of external trap potential, is illustrated in Fig. 1(c), where by means of increase of the magnetic field a periodic wave is transformed into a lattice of dark solitons.

In physical units, the cases depicted in Fig. 1 correspond to (a) $\mathcal{N} = 1.4 \times 10^4$ ^{7}Li atoms in a trap with $a_\perp \approx 7$ μm and $a_0 \approx 230$ μm, (b) $\mathcal{N} = 2 \times 10^4$ ^{7}Li atoms in a trap with $a_\perp \approx 6$ μm and $a_0 \approx 416$ μm, and (c) $\mathcal{N} = 10^4$ ^{23}Na atoms in a trap with $a_\perp = 3.4$ μm and $a_0 = 264$ μm. In the last case, however, one observes shifts of the soliton positions as well as a decrease of a density of particles located about the potential minimum because of weak oscillations of the condensate in the trap potential (the expanding phase is depicted in the figure). Thus, although effective τ corresponding to physical time [4] $t_0 = 40$ ms (used in all simulations) is not large, characteristic amplitudes of solitons placed in the center of the trap potential in the cases (a) and (b) match well the amplitude values following from the adiabatic approximation developed for a homogeneous NLS and for large τ, and in the case (c) one observes qualitative agreement. No instabilities of periodic waves are observed during periods of soliton train formation.

The developed analytical approach can be generalized to the NLS equation with linear damping; when the right-hand side of Eq. (2) is equal to $-i\gamma u$, γ being the damping constant, the substitution (3) yields again Eq. (4) but now with a modified value of ε: $\varepsilon \to \varepsilon - \gamma$. Hence, the equations for λ_1 and λ_2 hold their form with $z(t)$ defined as $z(t) = \ln g(t) - 2\gamma t$, so that the only change in Eqs. (11), (14), and (17) is multiplication of their right-hand sides by $\exp(-2\gamma t)$. Another generalization corresponds to moving soliton trains, which may be useful for treatment of BEC in ring traps. One of the applications of such moving soliton trains could be a "laser" of matter solitons.

The consideration provided above implies that one starts the adiabatic deformation with an initially periodic solution. A question arises about the possibility of creation of such a state experimentally. A natural approach to solving this problem would be the use of an optical trap [13]. In such a trap it is possible to create a nonlinear periodic distribution of a BEC [5,14]. Then, switching off the laser beams, producing the trap, will result in a periodic distribution of the condensate. However, it is not stable without the trap since it is not a solution of the respective GP equation. This difficulty can be overcome if simultaneously with switching off the optical trap one abruptly changes the scattering length (or alternatively provides a change of the number of particles) in such a way that the existing distribution will satisfy (2). To be more specific, let us consider an example of a BEC with a positive scattering length in an optical trap given by [14]: $V(x) = -2V_0 \mathrm{sn}(2\kappa x, m)$, where V_0 is the potential amplitude, $\kappa = \lambda_1 + \lambda_2$, and m is the same as in (17). The equation describing BEC evolution now admits a solution

$$u(x,t) = \sqrt{V_0 + (\lambda_1 - \lambda_2)^2}\, e^{2i(\lambda_1^2+\lambda_2^2)t} \mathrm{sn}[(\lambda_1 + \lambda_2)x, m].$$

This last function also solves (2) with $\sigma = 1$ and $g = [V_0 + (\lambda_1 - \lambda_2)^2]/(\lambda_1 - \lambda_2)^2$, and thus by switching off the potential $V(x)$ with simultaneously changing the scattering length by $\Delta a_s = a_0 V_0/(\lambda_1 - \lambda_2)^2$ one achieves the desired initial state. Notice that although experimentally a sn potential is not easily reachable in a general case, for a large range of m it is approximated very well by only a few first Fourier harmonics. For example, for a situation described in Fig. 1 one has $V(x) \approx -2V_0[1.47\sin(0.78x) + 0.15\sin(2.3x)]$ with the accuracy about 1%.

In order to estimate characteristic scales of adiabatic deformations we introduce an "aspect ratio" defined as $\delta = |\Delta A|/L$, where ΔA is a total variation of the amplitude of the periodic wave and L is its wavelength. Then the cases $\delta \gg 1$ and $\delta \ll 1$ correspond to a solitonic lattice and to a modulated plane wave. Dependence of the scattering length on time can be simulated by $g(t) = e^{t/t_0}$ (physical units). Taking as an example the solution depicted in Fig. 1(b), where $\delta \approx 1$ at $t = 0$, we find that already at $t \approx 30$ ms the aspect ratio becomes $\delta \approx 100$. The adiabaticity of the process means here that $\frac{\pi}{\omega_0} \gg t \gg \frac{\pi}{\omega_\perp}$ (physical units). It is satisfied well enough for traps with $\omega_\perp \gtrsim 2\pi \times 200$ Hz and $\omega_0 \lesssim 2\pi \times 5$ Hz. Stability of the above solutions has been studied numerically in [15], where it has been found that soliton trains are stable in the case of positive scattering length and are also stable in the case of negative scattering length for special choice of the parameters. In this context, the adiabatic variation of the scattering length, which results in the change of the wave parameters, can be used for stabilizing (or destabilizing) respective periodic solutions.

To conclude, we have outlined the main idea of management of periodic nonlinear waves in BEC's. The theory, although being developed for a homogeneous NLS equation, gives an accurate estimate of the central part of a BEC placed in a magnetic trap, the latter being studied numerically. The existence of trap can also be taken into account in the framework of a more sophisticated theory recently developed in [16].

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