Form Invariance of the Neutrino Mass Matrix

Ernest Ma
Physics Department, University of California, Riverside, California 92521, USA
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Consider the most general 3×3 Majorana neutrino mass matrix $\mathcal{M}$. Motivated by present neutrino-oscillation data, much theoretical effort is directed at reducing it to a specific texture in terms of a small number of parameters. This procedure is often *ad hoc*. I propose instead that for any $\mathcal{M}$ one may choose, it should satisfy the condition $U\mathcal{M}U^T = \mathcal{M}$, where $U \neq 1$ is a specific unitary matrix such that U^N represents a well-defined discrete symmetry in the $\nu_{e,\mu,\tau}$ basis, N being a particular integer not necessarily equal to 1. I illustrate this idea with a number of examples, including the realistic case of an inverted hierarchy of neutrino masses.

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Atmospheric neutrino oscillations have been firmly established [1] now for more than two years. Solar neutrino oscillations have also recently been confirmed [2]. The atmospheric mixing angle is maximal or nearly so with $\Delta m^2 \sim 2.5 \times 10^{-3}$ eV^2, whereas the solar mixing angle is not maximal but large ($\tan^2\theta \sim 0.45$) with two solutions for Δm^2, one on either side of 10^{-4} eV^2. Together, the neutrino mixing matrix is now determined to a very good first approximation by

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta/\sqrt{2} & \sin\theta/\sqrt{2} \\ -\sin\theta & \cos\theta/\sqrt{2} & \cos\theta/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}, \quad (1)$$

where $\nu_{1,2,3}$ are neutrino mass eigenstates. In the above, $\sin^2 2\theta_{\rm atm} = 1$ is already assumed and θ is the solar mixing angle. The U_{e3} entry has been assumed zero, but it is required only to be small [3], i.e., $|U_{e3}| < 0.16$.

It is the aim of much theoretical effort in the past several years [4] to find the correct neutrino mass matrix which will fit all the data. The starting point is usually the assumption that there are only three neutrinos and that they are Majorana fermions. The most general neutrino mass matrix in the basis $\nu_{e,\mu,\tau}$ (where the charged-lepton mass matrix is diagonal) is then of the form

$$\mathcal{M} = \begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix}, \quad (2)$$

where A, B, C may be chosen real by redefining the phases of $\nu_{e,\mu,\tau}$, but then D, E, F remain complex in general. Any model of neutrino mass (of which there are very many in the literature) always ends up with a simplification of $\mathcal{M}$, thereby reducing the number of independent parameters. The resulting form of $\mathcal{M}$ is of course always chosen to be consistent with experimental data, so that the model may be declared a success. This procedure is sometimes rather *ad hoc* and rife with arbitrary assumptions. Instead, I propose below a *novel* approach based on symmetry arguments.

Consider a *specific* unitary transformation U. Let $\nu_i' = U_{ij}\nu_j$, then $\mathcal{M}$ becomes $U\mathcal{M}U^T$ in the ν' basis. I propose that

$$U\mathcal{M}U^T = \mathcal{M} \quad (3)$$

be required as a condition on $\mathcal{M}$. If U represents a well-defined discrete symmetry, then this is nothing new. However, Eq. (3) also implies that

$$U^n\mathcal{M}(U^T)^n = \mathcal{M}, \quad (4)$$

where $n = 1, 2, 3$, etc. This sequence should terminate at $n = \bar{n}$ with $U^{\bar{n}} = 1$. Otherwise, the only possible solution for $\mathcal{M}$ would be a multiple of the identity matrix (in the case that U is also real). My proposal is that for a particular value $N < \bar{n}$, U^N should represent a well-defined discrete symmetry in the $\nu_{e,\mu,\tau}$ basis. Again if $N = 1$, there is nothing new. However, if $N \neq 1$, say, 2, then the unitary matrix U in Eq. (3) represents rather the "square root" of the discrete symmetry U^2. This is a new idea, with very interesting consequences as shown below.

Consider first the simple discrete symmetry

$$\nu_e \to \nu_e, \qquad \nu_{\mu,\tau} \to -\nu_{\mu,\tau}, \quad (5)$$

i.e.,

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (6)$$

The requirement of Eq. (3) fixes $D = E = 0$; thus

$$\mathcal{M} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & F \\ 0 & F & C \end{pmatrix}. \quad (7)$$

Now suppose instead that

$$U^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (8)$$

Then there are two obvious solutions for U, i.e.,

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \tag{9}$$

resulting in

$$\mathcal{M}_1 = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{M}_2 = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & F \\ 0 & F & 0 \end{pmatrix}. \tag{10}$$

Note that $\mathcal{M}_1$ and $\mathcal{M}_2$ are both special cases of the $\mathcal{M}$ of Eq. (7). Note also that $\mathcal{M}_2$ may be obtained in general with U_2 of the form

$$U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/n} & 0 \\ 0 & 0 & e^{-2\pi i/n} \end{pmatrix}, \tag{11}$$

where $n \geq 3$. This means that U_2 itself already represents a well-defined discrete symmetry in the $\nu_{e,\mu,\tau}$ basis, and there is nothing new about this application. On the other hand, neither $\mathcal{M}_1$ nor $\mathcal{M}_2$ are realistic candidates for the neutrino mass matrix.

Consider next the simple interchange discrete symmetry

$$\nu_e \rightarrow \nu_e, \qquad \nu_\mu \leftrightarrow \nu_\tau, \tag{12}$$

i.e.,

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{13}$$

The requirement of Eq. (3) fixes $D = E$ and $B = C$; thus

$$\mathcal{M} = \begin{pmatrix} A & D & D \\ D & B & F \\ D & F & B \end{pmatrix}. \tag{14}$$

This is now a very good candidate for a realistic neutrino mass matrix. In fact, if the four parameters A, B, D, F are chosen real, then this $\mathcal{M}$ is exactly diagonalized with Eq. (1). It is also the form advocated recently [5] as an all-purpose neutrino mass, where it is written as

$$\mathcal{M} = \begin{pmatrix} a+2b+2c & d & d \\ d & b & a+b \\ d & a+b & b \end{pmatrix}. \tag{15}$$

Depending on the actual values of a, b, c, d, this $\mathcal{M}$ was shown to have seven different solutions, three corresponding to the normal hierarchy, two to an inverted hierarchy, and two to three nearly degenerate neutrino masses. However, the symmetry of Eq. (12) cannot choose among these seven solutions.

Specific examples of Eq. (14) which have appeared in the literature include the cases $A = B + F$ [6], $A + D = B + F$ [7], and $A + B + F = 0$ [8]. It should also be pointed out that a *complete theory* exists for three nearly degenerate neutrino masses where the observed $\mathcal{M}_\nu$ is derived from a radiatively corrected [9] neutrino mass matrix based on the discrete symmetry A_4 [10]. In this model, the parameters b, c, d of Eq. (15) are generated in one-loop order by new physics at the TeV scale. This implies that the effective mass m_0 observed in neutrinoless double beta decay [11] should not be too small, or else the interpretation of $\Delta m^2 \sim 2.5 \times 10^{-3}$ eV^2 for atmospheric neutrino oscillations as a radiative correction becomes rather unnatural. With the recent data from the Wilkinson microwave anisotropy probe (WMAP), this mass also gets an upper bound [12] of 0.23 eV. The radiative A_4 model would require m_0 to be observable with some experimental improvement in either neutrinoless double beta decay or WMAP.

Now suppose instead that

$$U^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{16}$$

Then one obvious solution of its square root is

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-i)/2 & (1+i)/2 \\ 0 & (1+i)/2 & (1-i)/2 \end{pmatrix}, \tag{17}$$

resulting in

$$\mathcal{M}_1 = \begin{pmatrix} A & D & D \\ D & B & B \\ D & B & B \end{pmatrix} = \begin{pmatrix} 2b+2c & d & d \\ d & b & b \\ d & b & b \end{pmatrix}, \tag{18}$$

in the notations of Eqs. (14) and (15), where $F = B$ and $a = 0$ have now been fixed, respectively. The mass eigenvalues are then

$$m_{1,2} = 2b + c \mp \sqrt{c^2 + 2d^2}, \qquad m_3 = 0. \tag{19}$$

Since m_3 corresponds to the mass eigenstate $\nu_3 = (\nu_\mu - \nu_\tau)/\sqrt{2}$, this solution is an inverted hierarchy with

$$(\Delta m^2)_{\rm atm} \simeq (2b + c)^2 \simeq 4b^2, \tag{20}$$

$$(\Delta m^2)_{\rm sol} \simeq 4(2b + c)\sqrt{c^2 + 2d^2} \simeq 8b\sqrt{c^2 + 2d^2}. \tag{21}$$

Another solution is not so obvious, namely,

$$U_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \tag{22}$$

with $\omega = e^{2\pi i/3}$, resulting in

$$\mathcal{M}_2 = \begin{pmatrix} 2b+2d & d & d \\ d & b & b \\ d & b & b \end{pmatrix}, \tag{23}$$

which is a reduction of $\mathcal{M}_1$ of Eq. (18) by the condition $c = d$, thus predicting

$$\tan^2\theta = 2 - \sqrt{3} = 0.27, \tag{24}$$

which is marginally allowed by present experimental data at the low end of an acceptable range of values.

Whereas U^2 of Eq. (16) is the realization of the simple interchange discrete symmetry of Eq. (12), both U_1 of Eq. (17) and U_2 of Eq. (22) are not. Note, however, that the eigenvalues of U^2 are $(1, 1, -1)$, whereas those of U_1 and U_2 are $(1, 1, -i)$ and $(1, -1, +i)$, respectively.

Another possible choice of a simple discrete symmetry is

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (25)$$

which results in

$$\mathcal{M} = \begin{pmatrix} A & D & D \\ D & A & D \\ D & D & A \end{pmatrix}. \qquad (26)$$

Now

$$U^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (27)$$

also results in the same $\mathcal{M}$ and $U^3 = 1$ with the eigenvalues of both U and U^2 being $(1, \omega, \omega^2)$. However, this $\mathcal{M}$ is not a realistic candidate for the neutrino mass matrix.

Going back to Eq. (23), we see that d has to be much smaller than b to explain $(\Delta m^2)_{\rm sol} \ll (\Delta m^2)_{\rm atm}$. Suppose we now set $d = 0$; then $\mathcal{M}_2$ has a twofold degeneracy, i.e., $m_{1,2} = 2b$, $m_3 = 0$, with maximal $\nu_\mu - \nu_\tau$ mixing. This is then a very good starting point also for the understanding of solar neutrino oscillations in terms of an inverted hierarchy where $(\Delta m^2)_{\rm sol}$ and the solar mixing angle θ are radiative effects, in analogy to that of Ref. [9].

Consider thus the most general radiative corrections to $\mathcal{M}_\nu$, i.e.,

$$R = \begin{pmatrix} r_{ee} & r_{e\mu} & r_{e\tau} \\ r^*_{e\mu} & r_{\mu\mu} & r_{\mu\tau} \\ r^*_{e\tau} & r^*_{\mu\tau} & r_{\tau\tau} \end{pmatrix}; \qquad (28)$$

then

$$\mathcal{M}_\nu \to (1 + R)\mathcal{M}_\nu(1 + R^T) \qquad (29)$$

and becomes

$$m_0 \begin{pmatrix} 1 + 2r_{ee} & r^*_{e\mu} + (r_{e\mu} + r_{e\tau})/2 & r^*_{e\tau} + (r_{e\mu} + r_{e\tau})/2 \\ r^*_{e\mu} + (r_{e\mu} + r_{e\tau})/2 & (1 + 2r_{\mu\mu} + 2r_{\mu\tau})/2 & (1 + r_{\mu\mu} + r_{\tau\tau} + r_{\mu\tau} + r^*_{\mu\tau})/2 \\ r^*_{e\tau} + (r_{e\mu} + r_{e\tau})/2 & (1 + r_{\mu\mu} + r_{\tau\tau} + r_{\mu\tau} + r^*_{\mu\tau})/2 & (1 + 2r_{\tau\tau} + 2r^*_{\mu\tau})/2 \end{pmatrix}. \qquad (30)$$

Let

$$c \equiv r_{\mu\mu} + r_{\tau\tau} + 2{\rm Re}(r_{\mu\tau}) - 2r_{ee} > 0, \qquad (31)$$

$$d \equiv \sqrt{2}{\rm Re}(r_{e\mu} + r_{e\tau}); \qquad (32)$$

then the mass eigenvalues of the radiatively corrected $\mathcal{M}_\nu$ are

$$m_{1,2} = \left[1 + 2r_{ee} + \frac{c}{2} \mp \frac{1}{2}\sqrt{c^2 + 4d^2}\right]m_0, \quad m_3 = O(r^2)m_0, \qquad (33)$$

with the solar mixing angle θ given by

$$\tan^2\theta = 1 - \frac{2c}{\sqrt{c^2 + 4d^2} + c}, \qquad (34)$$

and

$$U_{e3} \simeq \frac{1}{\sqrt{2}}(r_{e\mu} - r_{e\tau}). \qquad (35)$$

In the standard model, $r_{ij} = 0$ for $i \neq j$, i.e., R is diagonal, hence $d = 0$ and even though m_1 and m_2 are split because $c \neq 0$, there is no mixing so ν_e does not oscillate at all. To obtain solar neutrino oscillations, we need flavor-changing interactions. As a simple example, consider the addition of three charged scalar singlets $\chi^+_{1,2,3}$ with the following interactions:

$$\mathcal{L}_{\rm int} = f[(\nu_1 l_3 - l_1\nu_3)\chi_1^+ + (\nu_2 l_3 - l_2\nu_3)\chi_2^+] + h(\nu_1 l_2 - l_1\nu_2)\chi_3^+ + {\rm H.c.} + m^2_{ij}\chi_i^+\chi_j^-, \qquad (36)$$

where $l_1 = e$, $l_{2,3} = (\mu \pm \tau)/\sqrt{2}$, and correspondingly for the neutrinos. This Lagrangian is invariant under the discrete symmetry

$$(\nu_1, l_1) \leftrightarrow (\nu_2, l_2), \qquad (\nu_3, l_3) \to (\nu_3, l_3), \qquad \chi_1^+ \leftrightarrow \chi_2^+, \qquad \chi_3^+ \to -\chi_3^+, \qquad (37)$$

which is broken softly by m^2_{ij}.

The radiative corrections r_{ij} are easily calculated in one loop as shown in Fig. 1. Note that c and d of Eqs. (31) and (32) are *finite* and derivable from the parameters of Eq. (36). In the inverted hierarchy of neutrino masses, $m_0 = \sqrt{(\Delta m^2)_{\rm atm}}$; hence

$$2\sqrt{c^2 + 4d^2} = \frac{(\Delta m^2)_{\rm sol}}{(\Delta m^2)_{\rm atm}} \simeq 0.04, \qquad (38)$$

and for $\tan^2\theta = 0.45$, $d/c = 1.22$. In this model, let $\chi_i = V_{ij}\chi'_j$, where χ'_j are mass eigenstates with masses m_j; then

$$c = \frac{f^2}{16\pi^2}\sum_{j=1}^{3}(|V_{2j}|^2 - |V_{1j}|^2)\ln m_j^2, \qquad (39)$$

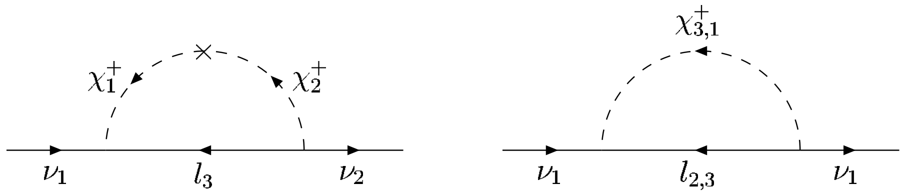

FIG. 1. Neutrino wave-function renormalizations.

$$d = \frac{f^2}{16\pi^2} \mathrm{Re} \sum_{j=1}^{3} V_{2j}^* V_{1j} \ln m_j^2, \qquad (40)$$

$$U_{e3} = \frac{-fh}{32\pi^2} \sum_{j=1}^{3} V_{2j}^* V_{3j} \ln m_j^2. \qquad (41)$$

Realistic values for c and d as well as a non-negligible complex U_{e3} are then possible if f and h are of order unity, and the m_j's are sufficiently different from one another.

Flavor-changing leptonic decays are predicted. For example, the amplitude for $\mu \to e\gamma$ is given by

$$\mathcal{A} = \frac{efm_\mu}{768\pi^2} \sum_{j=1}^{3} (fV_{1j}^* - hV_{3j}^*) \frac{V_{2j}}{m_j^2} \epsilon^\lambda q^\nu \bar{e} \sigma_{\lambda\nu} (1 + \gamma_5) \mu, \qquad (42)$$

whereas that of $\tau \to e\gamma$ is obtained by replacing m_μ by m_τ and h by $-h$. This means that one or the other of these decays may be suppressed but not both. Masses for χ_j^l of order 1 TeV are consistent with the present experimental upper bounds on the corresponding branching fractions.

In conclusion, a form invariance of the neutrino mass matrix has been proposed, i.e., $U\mathcal{M}_\nu U^T = \mathcal{M}_\nu$, where U is a specific unitary matrix and U^N (with N not necessarily equal to 1) represents a well-defined discrete symmetry in the $\nu_{e,\mu,\tau}$ basis. Using Eq. (12) as the definition of U^2, Eqs. (18) and (23) have been derived, allowing for an inverted hierarchy of neutrino masses, suitable for explaining the present data on solar and atmospheric neutrino oscillations. The possible radiative origin of $(\Delta m^2)_{\mathrm{sol}}$, $\tan^2\theta_{\mathrm{sol}}$, as well as U_{e3} has also been shown in a simple specific model with new flavor-changing interactions.

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