Universal Behavior of Crossover Scaling Functions for Continuous Phase Transitions

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We consider two different systems exhibiting a continuous phase transition into an absorbing state. Both models belong to the same universality class; i.e., they are characterized by the same scaling functions and the same critical exponents. Varying the range of interactions, we examine the crossover from the mean-field-like to the non-mean-field scaling behavior. A phenomenological scaling form is applied in order to describe the full crossover region, which spans several decades. Our results strongly support the hypothesis that the crossover function is universal.

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The critical behavior of a system exhibiting a second order phase transition with non-mean-like scaling behavior is strongly affected by the range of interactions. The longer the range of interactions, the stronger will be the critical fluctuations reduced. In the limit of infinite interactions, the system is characterized by the mean-field scaling behavior. But according to the well-known Ginzburg criterion [1], mean-field-like behavior occurs even for finite interaction ranges sufficiently far away from the critical point. A crossover to the non-mean-field scaling behavior takes place if one approaches the transition point. Although crossover phenomena are well understand in terms of competing fix points of the corresponding renormalization group approaches (see, for instance, [2]), some aspects of crossover phenomena are still open. For instance, it is an open question whether the so-called effective exponents fulfill certain scaling relations over the entire crossover region (see [3–6], and references therein). A second open question is the main theme of this Letter and addresses the universality of the crossover scaling functions. The range where the universal critical scaling behavior applies is usually restricted to a small vicinity around the critical point. Therefore, it is questioned that the full crossover region, which spans several decades in temperature or conjugated field, can be described in terms of universal scaling functions. Renormalization group approaches predicted a nonuniversal behavior if one uses finite cutoff lengths, whereas infinite cutoff lengths (which correspond to an unphysical vanishing molecular size) lead to a universal crossover behavior (see, for instance, [7,8]). On the other hand, the experimental situation is also unclear since measurements over the whole crossover region are difficult and accurate results are rare (see [4] for a short discussion). Thus, several attempts were performed in order to address this question via numerical simulations. For instance, the two- and three-dimensional Ising model with various interaction ranges is considered in a series of papers [4,5,9,10]. Using a sophisticated cluster algorithm for long-range interactions, it was possible to cover the full crossover region. In particular, a collapse of the susceptibility for different values of the interaction range was observed. Thus, the crossover can be described by a single scaling function in agreement with renormalization group approaches. But this result does not present evidence that this scaling function is universal, since only one system of a given universality class was considered.

The purpose of this Letter is to demonstrate via numerical simulations that the crossover from non-mean-field to mean-field-like scaling behavior can be represented by universal functions. We therefore consider two different systems exhibiting a continuous phase transition; both belong to the same universality class. The dynamics of the models is characterized by simple particle hopping processes, i.e., various interaction ranges can be easily implemented and highly accurate data are available. In this way it is possible to observe the full crossover region. Notice that we focus in our investigations on the particular universality class of absorbing phase transitions only for technical reasons. The demonstrated universality of crossover scaling functions can be applied to continuous phase transitions in general.

The first considered model is the so-called conserved lattice gas (CLG) which was introduced in [11]. In the CLG lattice sites may be empty or occupied by one particle. In order to mimic a repulsive interaction, a given particle is considered as active if at least one of its neighboring sites on the lattice is occupied by another particle. If all neighboring sites are empty, the particle remains inactive. Active particles are moved in the next update step to one of their empty nearest neighbor sites, selected at random.

The second model is the so-called conserved transfer threshold process (CTTP) [11]. Here lattice sites may be empty, occupied by one particle, or occupied by two particles. Empty and single occupied sites are considered as inactive, whereas double occupied lattice sites are considered as active. In the latter case, one tries to transfer both particles of a given active site to randomly chosen empty or single occupied nearest neighbor sites.

In our simulations (see [12,13] for details), we have used square lattices of linear size $L \leq 2048$. Every simulation starts from a random distribution of particles. After a transient regime both models reach a steady state characterized by the density of active sites ρ_a. The density ρ_a is the order parameter, and the particle density ρ is the control parameter of the absorbing phase transition; i.e., the order parameter vanishes at the critical density ρ_c according to $\rho_a \propto \delta\rho^{\beta}$, with the reduced control parameter $\delta\rho = \rho/\rho_c - 1$. In additional to the order parameter, we consider its fluctuations $\Delta\rho_a$. Approaching the transition point from above ($\delta\rho > 0$), the fluctuations diverge according to (see [12,13]) $\Delta\rho_a \propto \delta\rho^{-\gamma'}$. Below the critical density (in the absorbing phase), the order parameter as well as its fluctuations are zero in the steady state.

It was shown recently that the order parameter as well as its fluctuations obey the scaling forms [14]

$$\rho_a(\delta\rho, h) \sim \lambda^{-\beta}\tilde{R}(a_\rho \delta\rho \lambda, a_h h \lambda^{\sigma}), \tag{1}$$

$$a_\Delta \Delta\rho_a(\delta\rho, h) \sim \lambda^{\gamma'}\tilde{D}(a_\rho \delta\rho \lambda, a_h h \lambda^{\sigma}), \tag{2}$$

where h denotes an external field which is conjugated to the order parameter [12]. The universal scaling functions $\tilde{R}(x, y)$ and $\tilde{D}(x, y)$ are the same for all systems belonging to a given universality class, whereas all nonuniversal system-dependent features (e.g., the lattice structure, the update scheme, etc.) are contained in the so-called nonuniversal metric factors a_ρ, a_h, and a_Δ [15]. The universal scaling functions are normed by the conditions $\tilde{R}(1, 0) = \tilde{R}(0, 1) = \tilde{D}(0, 1) = 1$ and the nonuniversal metric factors can be determined from the amplitudes of

$$\rho_a(\delta\rho, h = 0) \sim (a_\rho \delta\rho)^{\beta}, \tag{3}$$

$$\rho_a(\delta\rho = 0, h) \sim (a_h h)^{\beta/\sigma}, \tag{4}$$

$$a_\Delta \Delta\rho_a(\delta\rho = 0, h) \sim (a_h h)^{-\gamma'/\sigma}. \tag{5}$$

These equations are obtained by choosing in the scaling forms [Eqs. (1) and (2)] $a_\rho \delta\rho \lambda = 1$ and $a_h h \lambda^{\sigma} = 1$, respectively.

Usually scaling functions are known only above the upper critical dimension D_c where the mean-field theory applies. In the case of the CLG model and CTTP, the mean-field scaling functions are given by [14,16] $\tilde{R}_{MF}(x, y) = x/2 + [y + (x/2)^2]^{1/2}$ as well as $\tilde{D}_{MF}(x, y) = \tilde{R}_{MF}(x, y)/[y + (x/2)^2]^{1/2}$, i.e., the mean-field exponents are $\beta_{MF} = 1$, $\sigma_{MF} = 2$, and $\gamma'_{MF} = 0$ (corresponding to a finite jump of the fluctuations). Below the upper critical dimension the universal scaling functions depend on the dimension and are unknown due to a lack of analytical solutions.

In the original CLG model and the original CTTP, particles of active sites are moved to nearest neighbors only; i.e., the range of interactions is $R = 1$. In the following, we consider a modified CLG model and a modified CTTP where particles of active sites are moved (according to the rules of each model) to randomly selected sites within a radius R. The order parameter is plotted in Fig. 1 for various ranges of interactions ($R \in \{1, 2, 4, \ldots, 128\}$). In the following, we examine how the varying interaction range affects the scaling behavior in the vicinity of the absorbing phase transition which now takes place at the critical density $\rho_{c,R}$.

The crossover scaling function at zero field has to incorporate the range of interactions as an additional scaling field. We make the phenomenological ansatz

$$\rho_a(\rho, R_{eff}) \sim \lambda^{-\beta_{MF}}\tilde{\mathfrak{R}}[\mathfrak{a}_\rho(\rho - \rho_{c,R})\lambda, \mathfrak{a}_R^{-1} R_{eff}^{-1} \lambda^{\phi}], \tag{6}$$

where the scaling function $\tilde{\mathfrak{R}}$ is universal since we allow for the nonuniversal metric factors $\mathfrak{a}_\rho$ and $\mathfrak{a}_R$. The Ginzburg criterion states that the mean-field picture is self-consistent in the active phase as long as the fluctuations within a correlation volume are small compared to the order parameter itself (see [17]). Thus, the crossover exponent is given by $\phi = (2\beta_{MF} - \nu_{MF} D)/D = (4 - D)/2D$, where $\nu_{MF} = 1/2$ denotes the critical exponents of the spatial correlation length. In order to avoid lattice effects, we use the effective interaction range [9]

$$R_{eff}^2 = \frac{1}{z}\sum_{i \neq j} |\underline{r}_i - \underline{r}_j|^2, \qquad |\underline{r}_i - \underline{r}_j| \leq R, \tag{7}$$

where z denotes the number of lattice sites within a radius R (see Table I). The mean-field scaling behavior should be recovered for $R \to \infty$, thus

$$\tilde{\mathfrak{R}}(x, 0) = \tilde{R}_{MF}(x, 0) = x^{\beta_{MF}}, \tag{8}$$

which implies $\mathfrak{a}_\rho = a_{\rho,R\to\infty}/\rho_{c,R\to\infty}$. These factors were already determined in previous works where absorbing phase transitions with infinite particle hopping were

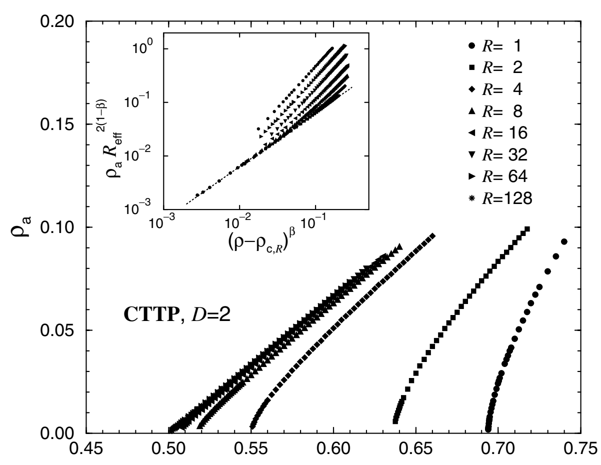

FIG. 1. The order parameter of the CTTP for various values of the interaction range R. With increasing interaction range, the critical density tends to the mean-field value $\rho_{c,R\to\infty} = 1/2$. The inset displays the order parameter which is rescaled according to Eq. (18). The dotted line corresponds to the scaling laws $y = mx$ with $m = (a_{\rho,R=1}/\rho_{c,R=1})^{\beta_{D=2}}$.

TABLE I. The range of interactions R, the corresponding number of next neighbors z on a square lattice, and the effective range of interactions $R_{\rm eff}$ for which we have carried out simulations. Additionally, the values of the critical densities are listed.

R	$z_{D=2}$	$R^2_{{\rm eff},D=2}$	$\rho_{{\rm c},R}^{\rm CLG}$	$\rho_{{\rm c},R}^{\rm CTTP}$
1	4	1	0.344 94(3)	0.693 92(1)
2	12	$\frac{7}{3}$	0.224 32(4)	0.636 49(2)
4	48	8	0.168 02(7)	0.550 05(3)
8	196	$\frac{1546}{49}$	0.140 50(9)	0.516 88(4)
16	796	$\frac{25\,274}{199}$	0.129 77(10)	0.505 52(6)
32	3208	$\frac{204\,875}{401}$	0.125 98(11)	0.501 61(7)
64	12 852	$\frac{13\,146\,247}{6426}$	0.124 99(16)	0.500 46(8)
128	51 432	$\frac{105\,255\,421}{12858}$	0.124 65(19)	0.500 19(9)

investigated [16]. The nonuniversal metric factor $\mathfrak{a}_R$ has to be determined by a second condition. Several ways are possible [e.g., $\tilde{\mathfrak{R}}(0,1)=1$] but for the sake of convenience we force $\tilde{\mathfrak{R}}$ to scale as

$$\tilde{\mathfrak{R}}(x,1) \sim x^{\beta_D}, \quad \text{for } x \to 0, \tag{9}$$

where β_D denotes the non-mean-field order parameter exponent of the corresponding D-dimensional system. Setting $\mathfrak{a}_R^{-1} R_{\rm eff}^{-1} \lambda^{\phi} = 1$ in Eq. (6) yields for zero field

$$\rho_{\rm a}(\rho, R_{\rm eff}) \sim (\mathfrak{a}_R R_{\rm eff})^{-\beta_{\rm MF}/\phi} \tilde{\mathfrak{R}}[\mathfrak{a}_\rho(\rho - \rho_{{\rm c},R}) \mathfrak{a}_R^{1/\phi} R_{\rm eff}^{1/\phi}, 1]. \tag{10}$$

Taking into account that the D-dimensional scaling behavior is recovered for $R = 1$, we find

$$\mathfrak{a}_R = \left(\frac{\rho_{{\rm c},R=1}}{a_{\rho,R=1}} \frac{a_{\rho,R\to\infty}}{\rho_{{\rm c},R\to\infty}} \right)^{\phi\beta_D/(\beta_{\rm MF}-\beta_D)}. \tag{11}$$

According to the above scaling form, we plot in Fig. 2 the rescaled order parameter $\rho_a(\mathfrak{a}_R R_{\rm eff})^2$ as a function of the rescaled control parameter $\mathfrak{a}_\rho(\rho - \rho_{{\rm c},R})(\mathfrak{a}_R R_{\rm eff})^2$ for the two-dimensional ($\phi = 1/2$) CLG model and the two-dimensional CTTP. The values of the metric factors are listed in Table II and are determined from data of previous simulations (via a direct measurement of the amplitudes of the corresponding power laws). Thus, no parameter fitting is applied. We observe an excellent data collapse for the entire range of the crossover, confirming the phenomenological ansatz. In the inset in Fig. 2 we plot the same data without metric factors. As can be seen, each model is characterized by its own scaling function.

Since the entire crossover region covered several decades, it could be difficult to observe small but systematic differences between the scaling functions of both models. It is therefore instructive to examine the crossover via the so-called effective exponent [5]

$$\beta_{\rm eff} = \frac{\partial}{\partial \ln x} \ln \tilde{\mathfrak{R}}(x,1). \tag{12}$$

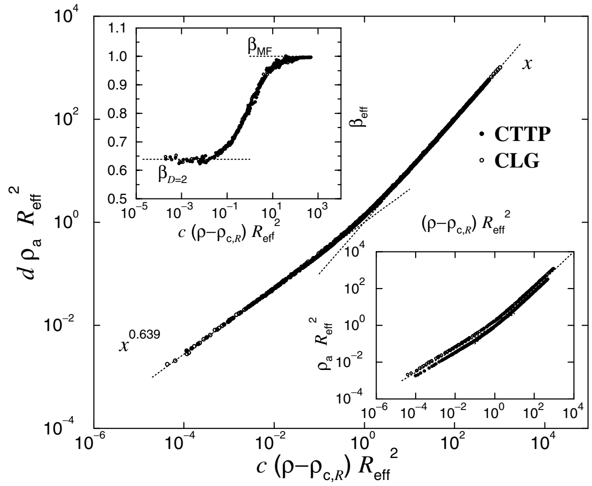

FIG. 2. The rescaled order parameter. The metric factors are given by $c = \mathfrak{a}_\rho \mathfrak{a}_R^2$ and $d = \mathfrak{a}_R^2$. The data of both models display an excellent collapse to the universal crossover scaling function $\tilde{\mathfrak{R}}(x,1)$. The dashed lines correspond to the asymptotic behavior of the two-dimensional system ($\beta_{D=2} = 0.639$ [13]) and of the mean-field behavior ($\beta_{\rm MF} = 1$). Neglecting the metric factors, each model is characterized by its own scaling function (see lower right inset). The upper left inset displays the effective exponents $\beta_{\rm eff}$ for both considered models. The data of the CLG model and of the CTTP exhibit an excellent data collapse.

The corresponding data are shown in Fig. 2. The excellent data collapse of $\beta_{\rm eff}$ of both models over more than seven decades strongly supports the hypothesis that the crossover function is a universal function.

We now consider the order parameter fluctuations. Analogous to the order parameter, we make the scaling ansatz ($\gamma'_{\rm MF} = 0$)

$$\mathfrak{a}_\Delta \Delta\rho_{\rm a}(\rho, R_{\rm eff}) \sim \tilde{\mathfrak{D}}[\mathfrak{a}_\rho(\rho - \rho_{{\rm c},R})\lambda, \mathfrak{a}_R^{-1} R_{\rm eff}^{-1} \lambda^{\phi}]. \tag{13}$$

Again, the mean-field behavior should be recovered for $R \to \infty$, implying $\tilde{\mathfrak{D}}(x,0) = \tilde{D}_{\rm MF}(x,0) = 2$ as well as $\mathfrak{a}_\Delta = a_{\Delta,R\to\infty}$. Setting $\mathfrak{a}_R^{-1} R_{\rm eff}^{-1} \lambda^{\phi} = 1$ yields

$$\mathfrak{a}_\Delta \Delta\rho_{\rm a}(\rho, R_{\rm eff}) \sim \tilde{\mathfrak{D}}[\mathfrak{a}_\rho(\rho - \rho_{{\rm c},R}) \mathfrak{a}_R^{1/\phi} R_{\rm eff}^{1/\phi}, 1]. \tag{14}$$

For finite R, the fluctuations diverge at the critical point; i.e., the universal function $\tilde{\mathfrak{D}}$ scales as

$$\tilde{\mathfrak{D}}(x,1) \sim m_{\Delta,\rho} x^{-\gamma'_D}, \quad \text{for } x \to 0. \tag{15}$$

The universal amplitude $m_{\Delta,\rho}$ can be determined in the

TABLE II. The nonuniversal metric factors determined from previous simulations via direct measurements of the corresponding power laws.

	$\rho_{{\rm c},R=1}$	$a_{\rho,R=1}$	$a_{\Delta,R=1}$	$\rho_{{\rm c},R\to\infty}$	$a_{\rho,R\to\infty}$	$a_{\Delta,R\to\infty}$
$\text{CLG}_{D=2}$	0.344 94	0.5089	15.50	0.1244	0.1635	12.02
$\text{CTTP}_{D=2}$	0.693 92	0.3410	50.18	1/2	0.3345	24.85

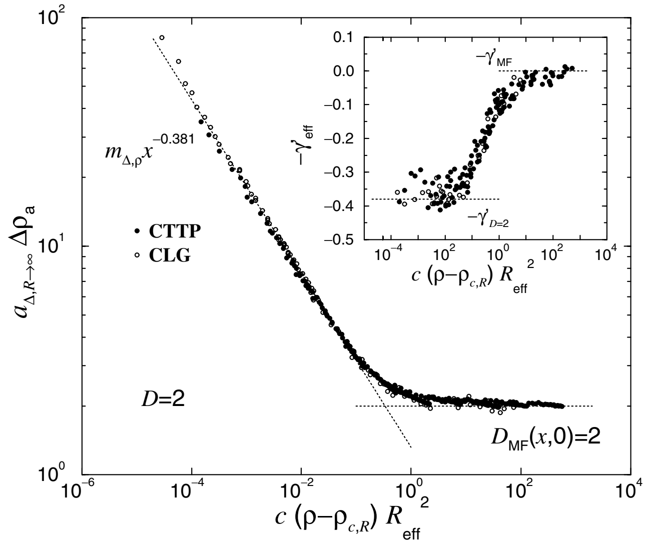

FIG. 3. The rescaled fluctuations of the order parameter. The metric factor is given by $c = \mathfrak{a}_\rho \mathfrak{a}_R^2$. The data of both models display a good collapse to the universal crossover scaling function $\tilde{\mathfrak{D}}(x, 1)$. The dashed lines correspond to the asymptotic behavior of the two-dimensional system ($\gamma'_{D=2} = 0.381$ [13] and $m_{\Delta,\rho} = 1.28$) and of the mean-field behavior ($\gamma'_{\rm MF} = 0$). The inset displays the corresponding effective exponent $\gamma'_{\rm eff}$.

following way: The scaling form Eq. (13) has to equal for $R = 1$ the D-dimensional scaling behavior [see Eq. (2)]

$$\Delta\rho_{\rm a} \sim a_{\Delta,R=1}^{-1}\tilde{D}(1,0)\left(a_{\rho,R=1}\frac{(\rho-\rho_{{\rm c},R=1})}{\rho_{{\rm c},R=1}}\right)^{-\gamma'_D}. \quad (16)$$

Thus, we find

$$m_{\Delta,\rho} = \tilde{D}(1,0)\frac{a_{\Delta,R\to\infty}}{a_{\Delta,R=1}}\left(\frac{\rho_{{\rm c},R=1}}{a_{\rho,R=1}}\frac{a_{\rho,R\to\infty}}{\rho_{{\rm c},R\to\infty}}\right)^{\gamma'_D\beta_{\rm MF}/\beta_{\rm MF}-\beta_D}, \quad (17)$$

where the value of the universal scaling function $\tilde{D}(1,0) = 1.87 \pm 0.11$ is obtained via direct measurements of the corresponding two-dimensional systems. According to the scaling form Eq. (14), we plot in Fig. 3 the rescaled fluctuations as a function of the rescaled control parameter for the two-dimensional CLG model as well as for the CTTP. We observe again a good collapse of the data over the entire region of the crossover. Furthermore, both asymptotic behaviors are recovered, confirming the scaling ansatz Eq. (13).

The corresponding effective exponent $\gamma'_{\rm eff} = \partial\ln\tilde{\mathfrak{D}}(x,1)/\partial\ln x$ is displayed in the inset in Fig. 3. Although the data of the effective exponent are suffering from statistical fluctuations, one can see that both models are characterized by the same universal behavior.

At the end, we consider the critical amplitudes of the scaling functions. Using the above discussed scaling forms, it easy to show that these amplitudes display a singular dependence on the range of interactions. For instance, the order parameter scales sufficiently close to the transition point ($x \to 0$) as [see Eqs. (9)–(11)]

$$\rho_{\rm a}(\rho, R_{\rm eff}) \sim R_{\rm eff}^{(\beta_D-\beta_{\rm MF})/\phi}\left(a_{\rho,R=1}\frac{\rho-\rho_{{\rm c},R}}{\rho_{{\rm c},R=1}}\right)^{\beta_D}. \quad (18)$$

Thus, this scaling law and the corresponding scaling law for the fluctuations are valid only for finite interaction ranges, whereas they become useless for infinite R, signaling the change in the universality class for $R \to \infty$. This amplitude scaling can be observed in simulations. The inset in Fig. 1 shows the corresponding data for the CTTP. As can be seen, the data of various interaction ranges tend to the same power-law behavior if one approaches the transition point.

In conclusion, the crossover from mean-field to non-mean-field scaling behavior is numerically investigated for two different models exhibiting an absorbing phase transition. Increasing the range of interactions, we are able to cover the full crossover region which spans several decades of the control parameter. The excellent collapse of the effective exponents of both models strongly supports the interpretation of the crossover scaling functions in terms of universality; i.e., the crossover function is universal.

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