Static Criteria for the Existence of Coulomb Strings in Storage Rings

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We derive four rigorous conditions for the stability of Coulomb strings in circular storage rings. These criteria are well met by the existing data from experiments in SIS, ESR, and CRYring but not by the NAP-M experiment. We calculate the potential of the joint transverse zigzag excitation and the longitudinal motion against each other of a string of charged particles as a function of their amplitudes and with the linear density as parameter. This potential exhibits a saddle point in amplitude space which, if overcome, destroys the order of the string. The conditions of stability are derived from the position and height of the saddle point which are fairly independent of the linear density. Our findings confirm the supposition that only the Coulomb interaction in the immediate vicinity of very close encounters of particles is important for the existence of strings.

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Laser cooled singly charged ions in ion traps crystallize at very low temperature. For recent developments in this field, see [1]. During the time of their first experimental observation [2] Schiffer and Kienle [3] suggested looking for such Coulomb crystals in heavy ion beams as well. This should be possible since the high charge of fully stripped ions enhance the plasma parameter Γ, the ratio between average mutual Coulomb energy and thermal energy, substantially by orders of magnitude despite the fact that the achievable temperature is much higher with electron cooling. The infinitely long cylindrical structures obtained in elaborate simulations with constant radial focusing by Rahman and Schiffer [4] and lateron [5] range from 1D linear strings over zigzags at low density, over helices to 3D multishell cylinders at high density. Such objects are called crystalline beams. By virtue of their high luminosity they are of great experimental interest. However, application of realistic alternate gradient quadrupole focusing and, in particular, of dipole bending forces in the simulations destroyed the eventual crystalline structures. It was only Wei *et al.* [6] who then worked out conditions for the lattice under which they might survive. Machines with such lattices, however, do not exist and *3D crystalline beams* have not yet been generated in heavy ion accelerators or storage rings. Based on these efforts, attempts have been made [7,8] to derive stability conditions from the critical plasma parameter $\Gamma > 178$ [9] where the 3D one component plasma (OCP) crystallizes, resulting in order of magnitude estimates.

On the other hand, if one restricts itself to *linear 1D strings,* the destructive bending forces cannot act and simple 1D ordering is feasible. Such ordering is of liquid type where only the two nearest neighbors interact, all other particles being only spectators. Furthermore, since the ions are only at most a few micrometers off the axis, the difference between constant and alternate gradient focusing plays only a minor role. Such linear ordering has been verified in recent years. Schottky measurements of the momentum spread $\delta p/p$ at beams of various species of highly charged and extremely electron cooled heavy ions at very low density in the storage ring ESR [10] and in the synchroton SIS [11] of GSI have revealed that below a certain threshold in density intrabeam scattering ceases to act. Similar observations have been made in the CRYring [12] of Stockholm. Also, the fluorescence signal emitted from ordered strings of laser cooled singly charged magnesium ions have been seen directly in the rf quadrupole storage ring PALLAS at München [13]. The beam radius below the critical threshold of stable strings was determined with the help of a scraper to be smaller than some micrometers, and the corresponding longitudinal and transverse kinetic energies are estimated to be of the order of some ten or hundred millielectronvolts, respectively.

The typical drop from $\delta p/p \approx 5 \times 10^{-6}$ by an order of magnitude down to values close to 5×10^{-7} at average interparticle distances d of the order of centimeters has been explained as the ions form a linear string where the particles in the comoving frame of reference move against each other but cannot pass any more [14]. Here classical Monte Carlo trajectory calculations were performed with charged particles heading towards each other under constant focusing with the betatron frequency ω_β of the respective ring and with the experimental longitudinal and transverse thermal energies. As results, the probability of reflection (or transmission) increases (drops) sharply when going to larger average interparticle distances. Recently, even more elaborate calculations with the full lattice of the circular accelerators were performed by Okamoto and co-workers [15], yielding similar results.

As this model can *explain* the given experimental data, it cannot yet *predict* the linear density λ and longitudinal, $T_{\|}$, and transverse, $T_{\perp}$, thermal energies at which this effect shows up. The only hitherto well established necessary relation for the existence–but not the stability–of

Coulomb strings is

$$d > 1.4a_{WS}, \tag{1}$$

where d is the average interparticle distance along the beam direction. It stems from the transition density where at zero temperature a string ceases to be stable and turns into a zigzag configuration at $\lambda < 0.709$, where $\lambda = a_{WS}/d$ is the linear density introduced in Ref. [5] and a_{WS} is the Wigner-Seitz radius

$$a_{WS} = \left(\frac{3q^2}{2M\omega_\beta^2}\right)^{1/3}, \tag{2}$$

with $q = Ze$ being the charge, Z the charge number, M the mass of the ions, and $4\pi\epsilon_0 = 1$ for convenience. It is the purpose of this Letter to derive rigorous criteria for the thermal energies under which the Coulomb strings are stable, not based on the 3D OCP but on the excitations of strings.

Suppose that at rest the ions are distributed equidistantly along the z axis, $z_j = jd$, ($j = -\infty, \ldots -1, 0, 1, \ldots, \infty$) with radii $r_j = 0$. This string can vibrate in different cosinelike (even parity $\Pi = +$) or sinelike (odd parity $\Pi = -$) modes M^Π longitudinally as well as transversely [16],

$$z_j = jd + ud\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(jq), \qquad r_j = \rho d\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(jq), \qquad \text{for } \Pi = \begin{Bmatrix}+\\ -\end{Bmatrix}. \tag{3}$$

Here q is the wave number related to the mode number by $M = 2\pi/q$. In general, the longitudinal and transverse modes are decoupled. Since higher modes have higher energies and since we are only interested in the lowest possible excitations, we restrict ourselves to the lowest 2^+ mode in the longitudinal direction, which is the zigzag mode. In this case the dimensionless potential energy per particle in units of q^2/d reads

$$\Phi(v, \rho) = \frac{3\rho^2}{4\lambda^3} + \frac{1}{2}\sum_{j=1,3,\ldots}^{\infty}\{[j^2 + 4(\rho^2 + v^2 + 4vj)]^{-1/2} + [j^2 + 4(\rho^2 + v^2 - 4vj)]^{-1/2} - 2/j\}, \tag{4}$$

where $v = u/d$ and $\rho = r/d$. The first term of Eq. (4) is just the harmonic radial confining potential. A contour plot of this potential for the example $\lambda = 0.25$ is shown in Fig. 1. Here the lower left-hand corner and, by periodicity, also the lower right-hand corner correspond to the string being at rest. Upwards, the system needs energy to work against the radial harmonic confining potential.

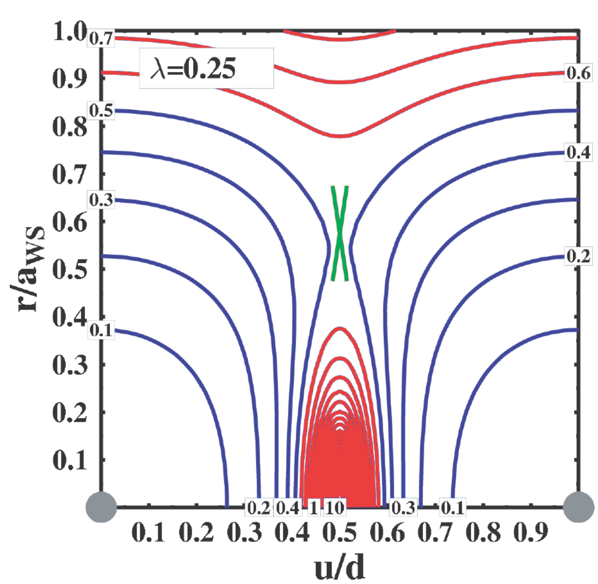

FIG. 1 (color online). Potential energy Eq. (4) of the collective zigzag excitation. The numbers in boxes label the contour lines in units of q^2/a_{WS}. The ground states are indicated by the light dots and the saddle point is marked by a cross. Note the different units in the longitudinal and transverse directions.

Going to the right (or left) from the outer corners means that the ions approach each other with $u = d/2$ being at closest contact. There, without radial excursion ($r = 0$) the potential exhibits an infinitely high mountain. For an intermediate radial amplitude, however, the potential has a saddle point at $u = d/2$ which can be overcome if the longitudinal kinetic energy is sufficiently high.

The location and height of this saddle point is shown in Fig. 2 as a function of the linear density. Note that the saddle point of the zigzag mode vanishes by definition at the critical $\lambda = 0.709$ where the ground state string turns into a zigzag. It can be seen that the radial position in units of a_{WS} is almost constant in a wide range of small densities. The same applies to its height in units of

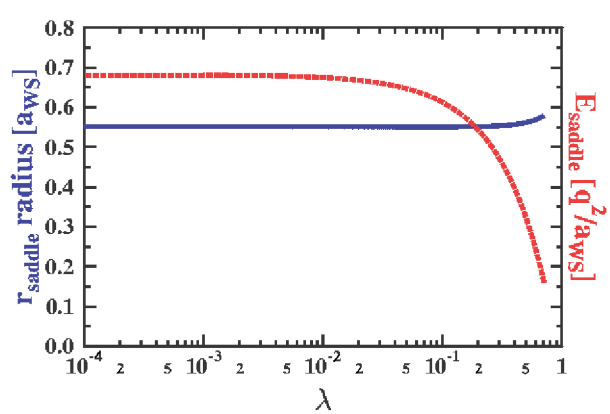

FIG. 2 (color online). Saddle point radius in units of a_{WS} (solid line) and saddle point energy in units of q^2/a_{WS} (dashed line) on the same scale.

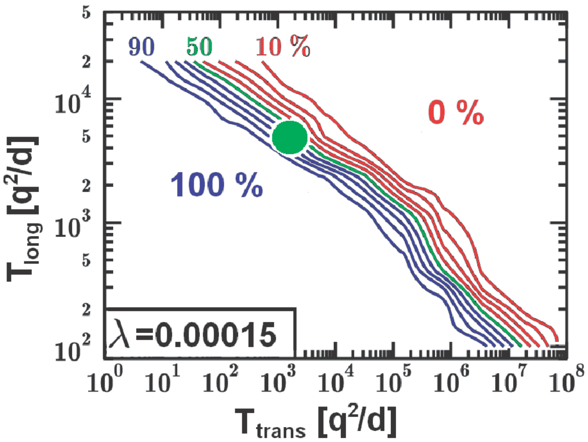

FIG. 3 (color online). Contour plot of the reflection probability versus kinetic energies in units of q^2/d (after Ref. [14]). The dot indicates the result of the criteria (5) and (6).

q^2/a_{WS}. As a result we derive the following conditions for the stability of Coulomb strings in circular rings:

$$r \lesssim 0.6 a_{WS} \quad \text{or} \quad T_\perp \lesssim 0.25 q^2/a_{WS}, \tag{5}$$

$$T_{||} \lesssim 0.7 q^2/a_{WS}. \tag{6}$$

Hereby the second part of Eq. (5) obtains from $T_\perp = M\omega_\beta^2 r^2/2$ without taking into account fine details such as the difference between maximum and rms radius. The longitudinal kinetic energy can be converted to momentum spread with the help of the relation $T_{||} = M(\beta c \delta p/p)^2/(8/\ln 2)$ [10], where βc is the beam velocity. Note that the relations (5) and (6) contain as measures of distance just the Wigner-Seitz radius, a combination of beam properties, and not the interparticle distance or the beam density. This reflects the fact that most of the time the ions do not interact notably. Only if they come close within a distance of the order of a_{WS} they feel the repulsive Coulomb potential.

Another relation is derived from the stiffness of the potential of Fig. 1 at the ground state ($r = 0$, $u = 0$) marked by the light dot in the direction of the saddle point. Equating the saddle height $0.7q^2/a_{WS}$ to $M\omega_{||}^2 r_{\text{saddle}}^2/2$, one obtains the Coulomb period $\tau_{||} = 2\pi/\omega_{||}$. Thus the ratio of Coulomb period to betatron period must be

$$\tau_{||}/\tau_\beta \gtrsim \lambda^{-1}, \tag{7}$$

reflecting the fact that the period of an average Coulomb scattering must be large as compared to a single betatron period. This condition is well fulfilled for the experiments listed below.

As a comparison with the results derived from the molecular dynamics calculations of Ref. [14] is shown in Fig. 3 at the typical linear density $\lambda = 0.000\,15$. Here the contour lines label the reflection probability which rises sharply from 0 to 100% as crossing the threshold. The resulting point from the criteria (5) and (6) is indicated by the big dot, right in the center. Another comparison is supplied by Table I where all data of existing experiments are listed. Here $0.7q^2/a_{WS}$ is of the same order of magnitude as $T_{||}$. However, the experimentally determined rms radii or the transverse kinetic energies, are larger by factors of 2 to 5. A newer analysis of the U^{92+} machine experiment in the ESR [17] gave a horizontal radius of as small as 5 μm, which is even smaller than our criterion. Note also that the potential energy landscape of Fig. 1 is very flat above and below the saddle point and that the reflection probability of Fig. 3 goes from 0 to 100% within 2 orders of magnitude in $T_\perp$, thus leaving enough room for much larger radii and kinetic energies. The large particle density of the NAP-M proton experiment at Novosibirsk of 1976 [18] clearly violates the first criterion (1), and the very short Coulomb period

TABLE I. Experimental data, momentum spread $\delta p/p$, average particle distance d, Wigner-Seitz radius a_{WS}, unit of energy q^2/a_{WS}, linear density λ, longitudinal and transverse temperatures $T_{||}$, $T_\perp$, rms radius r_{rms} and scattering ratio (after Ref. [14]).

Ring	Ion	$\delta p/p$ [10^{-6}]	d [cm]	a_{WS} [μm]	q^2/a_{WS} [meV]	λ	$T_{||}$ [meV]	$T_\perp$ [meV]	r_{rms} [μm]	$\tau_{||}/\tau_\beta$
ESR	$^{12}C^{6+}$	2	0.17	7.7	6.7	0.0046	1.5	90	30	450
ESR	$^{20}Ne^{10+}$	2	0.25	9.1	16	0.0036	10	150	30	700
ESR	$^{40}Ar^{18+}$	4	4	8.9	52	0.000 20	19	300	21	8500
ESR	$^{48}Ti^{22+}$	2.5	0.44	11.5	61	0.0026	9	370	30	950
ESR	$^{58}Ni^{28+}$	4	8	13.6	83	0.000 16	26	440	33	1300
ESR	$^{86}Kr^{36+}$	4	6	13.3	140	0.00022	39	640	30	8000
ESR	$^{132}Xe^{54+}$	6	10	15.0	280	0.000 15	120	1000	30	10 000
ESR	$^{197}Au^{79+}$	6	2	14.0	640	0.000 70	290	1500	21	2200
ESR	$^{238}U^{92+}$	5	10	14.6	830	0.000 15	240	1800	21	13 000
ESR	p	1	0.2	9.1	0.16	0.0045	0.01	7.5	70	1000
SIS	$^{86}Kr^{36+}$	15	2	50	37	0.000 25	40	100	60	2000
CRYring	$^{64}Ni^{17+}$	5	1	19.8	21	0.0019	2.3	···	···	1000
PALLAS	24Mg$^+$	≈ 40	20 μm	7.8	0.18 μeV	0.39	0.26 μeV	$\ll 30$	2	100
NAP-M	p	?1	2 μm	8.0	0.18	4.2	?0.01	?25	150	0.6

violates the fourth criterion, thus indicating that order could not have been reached.

In summary, we have stated four criteria for the stability of Coulomb strings in circular machines by employing the collective properties of charged strings. No reference was necessary to the properties of the three-dimensional one component plasma. The simple estimates agree well with the results from the more elaborate calculations of the reflection probabilities. Although the criteria give rigorous upper limits for the *kinetic energies* (or *temperatures*) it cannot predict the critical *densities* where the strings start or cease to exist. This, in turn, is determined by the dynamical process of the cooling force counteracting intrabeam scattering which is out of scope of our static model but is subject of work under way.

The idea of this work originated after a seminar at GSI of Dieter Möhl who reported on calculations of the luminosity of an electron-ion collider with an ordered ion beam at RIKEN [7]. The author thanks him for fruitful discussions.

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