Traversable Wormholes with Arbitrarily Small Energy Condition Violations

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Traversable wormholes necessarily require violations of the averaged null energy condition, this being the definition of "exotic matter." However, the theorems which guarantee the energy condition violation are remarkably silent when it comes to making quantitative statements regarding the "total amount" of energy condition violating matter in the spacetime. We develop a suitable measure for quantifying this notion and demonstrate the existence of spacetime geometries containing traversable wormholes that are supported by arbitrarily small quantities of exotic matter.

Introduction.—We know that traversable wormholes require "exotic matter," that is, violations of the averaged null energy condition [ANEC] [1–4]. But, do we need "large" amounts of ANEC-violating matter or are "small" amounts sufficient to do the job? This question is particularly interesting in view of the fact that quantum effects are known to induce *some* energy condition violations [5]. Furthermore numerical evidence suggests that quantum effects may be sufficient to support a wormhole throat [6]. On the other hand, there are significant limitations (the Ford-Roman "quantum inequalities" and their variants) on the "size" and "distribution" of quantum-induced energy condition violations [7].

To set the stage, consider the four great results of global analysis in classical general relativity—the area-increase, positive mass, singularity, and topological censorship theorems. Some of these classical results seem to survive the introduction of quantum physics, while others do not.

In the case of the area-increase theorem (see, e.g., [8]) we know that quantum effects, though extremely tiny, violate the ANEC (and other energy conditions) and are a necessary precondition for Hawking radiation [9]. The effect though tiny is *secular*, so it accumulates and completely reverses the conclusion of the classical area-increase theorem once quantum effects are brought into account.

In contrast, for the positive mass theorem [10] the situation is rather different. For a finite spherically symmetric static distribution we know (using curvature coordinates) that

$$m_\infty = m_0 + \int_{\max\{0, 2m_0\}}^\infty 4\pi r^2 \rho(r) dr. \qquad (1)$$

So if the asymptotic mass is to be large and negative, then similarly either the central "bare" mass m_0 or the integrated density must be large and negative. You cannot get large negative asymptotic mass from infinitesimal weak energy condition (WEC) violations.

The situation with respect to the singularity theorems (see, e.g., [8]) is intermediate. The various ways currently known of violating the energy conditions [11] lead to "technical" violation of the singularity theorems—the hypotheses of the singularity theorems (as currently formulated) are not satisfied by empirical reality. Whether it is possible to avoid singularities armed only with arbitrarily small quantities of ANEC-violating matter is not presently known.

With regard to topological censorship, the key question is this: Is the existence of traversable wormholes more akin to the situation with the area-increase theorem (infinitesimal ANEC violations reverse the conclusion of the theorem) or is it more akin to the situation with regard to positive mass (large WEC violations needed for a large negative mass)? We shall answer this question by explicit example; we work with static spherically symmetric spacetimes and develop a suitable quantitative measure of the total amount of exotic matter. We then display a particular class of spacetime geometries that contain a traversable wormhole (and thereby violate topological censorship), but which are supported by arbitrarily small quantities of exotic matter.

We conclude that the topological censorship theorem is more akin to the area-increase theorem than to the positive mass theorem—small violations of the energy conditions are sufficient to evade the conclusions of the theorem.

Constructing the wormhole.—Consider a static spherically symmetric spacetime and go to Schwarzschild

coordinates (curvature coordinates) [1,4]:

$$ds^2 = -\exp[2\phi(r)]dt^2 + \frac{dr^2}{1 - b(r)/r}$$
$$+ r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2}$$

Then, using the Einstein field equations, the components of the diagonal energy-momentum tensor in an orthonormal basis turn out to be (units: $G = c = 1$) [1,4]

$$\rho = \frac{1}{8\pi}\frac{b'}{r^2}; \qquad p_r = \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\left\{1 - \frac{b}{r}\right\}\frac{\phi'}{r}\right], \tag{3}$$

$$p_t = \frac{1}{8\pi}\left\{1 - \frac{b}{r}\right\}\left[\phi'' + \phi'\left(\phi' + \frac{1}{r}\right)\right]$$
$$- \frac{1}{2}\left(\frac{b}{r}\right)'\left(\phi' + \frac{1}{r}\right), \tag{4}$$

where ρ, p_r, and p_t are the energy density, the radial, and tangential pressures, respectively. If the equation $b(r) = r$ has a nontrivial solution r_0 and $\exp[\phi(r_0)] \neq 0$, then we can cut the spacetime at $r = r_0$ and paste it onto a second copy of itself, *with a C^2 geometry at $r = r_0$*. The extrinsic curvature of the hypersurface $r = r_0$ (η is the proper radial distance) is

$$K_{ab} \propto \frac{\partial h_{ab}}{\partial \eta} = \frac{\partial h_{ab}}{\partial r}\frac{\partial r}{\partial \eta} = \frac{\partial h_{ab}}{\partial r}\sqrt{1 - b(r)/r} \to 0. \tag{5}$$

As long as $\exp[\phi(r_0)] \neq 0$ you can go to an orthonormal basis and have $K_{\hat{a}\hat{b}} = 0$. (Hatted indices denote components in an orthonormal basis.) The junction condition formalism [12] now guarantees the geometry is C^2 across the gluing hypersurface. (Under normal conditions the junction formalism yields a C^1 geometry; it is the vanishing of the extrinsic curvature at the junction that in this case makes the geometry C^2.)

The ANEC integral along a radial null geodesic is

$$I = \oint[\rho + p_r]\exp[-2\phi]d\lambda = \oint[\rho + p_r]\exp[-\phi]d\eta. \tag{6}$$

An integration by parts yields [4]

$$I = -\frac{1}{4\pi}\oint \frac{1}{r^2}e^{-\phi}\sqrt{1 - \frac{b}{r}}dr < 0. \tag{7}$$

Unfortunately this is a line integral, with dimensions (mass)/(area), not a volume integral, and so gives no useful information regarding the total amount of energy condition violating matter.

The basic volume-integral theorem relates the asymptotic mass to the throat radius r_0 and the density by

$$m_\infty = \frac{r_0}{2} + \int_{r_0}^\infty 4\pi r^2 \rho(r)dr. \tag{8}$$

This is a simple generalization of the ordinary mass

formula for relativistic stars to traversable wormholes [13]. Now

$$\int dV = \int 4\pi r^2 dr; \qquad \oint dV = 2\int_{r_0}^\infty 4\pi r^2 dr \tag{9}$$

is a very natural integration measure in Schwarzschild coordinates. This is the measure that appears in the simplest formula for the total mass. Indeed, including both asymptotic regions, we have

$$\oint \rho dV = 2m_\infty - r_0. \tag{10}$$

We now develop our key volume-integral result, using this *same* "r^2dr" measure. It is easy to check that

$$\rho + p_r = \frac{1}{8\pi r}\left\{1 - \frac{b}{r}\right\}\left[\ln\left(\frac{\exp[2\phi]}{1 - b/r}\right)\right]'. \tag{11}$$

Then integrating by parts

$$\oint[\rho + p_r]dV = \left[(r - b)\ln\left(\frac{\exp[2\phi]}{1 - b/r}\right)\right]_{r_0}^\infty$$
$$- \int_{r_0}^\infty(1 - b')\left[\ln\left(\frac{\exp[2\phi]}{1 - b/r}\right)\right]dr. \tag{12}$$

The boundary term at r_0 vanishes by our construction of the wormhole. The boundary term at infinity vanishes because of the assumed asymptotic behavior. Then

$$\oint[\rho + p_r]dV = -\int_{r_0}^\infty(1 - b')\left[\ln\left(\frac{\exp[2\phi]}{1 - b/r}\right)\right]dr. \tag{13}$$

This volume-integral theorem provides information about the total amount of ANEC-violating matter in the spacetime [14]. For the transverse pressure we have

$$p_t = p_r + \frac{r}{2}\{p_r' + (\rho + p_r)\phi'\}, \tag{14}$$

but in the general case this does not lead to a particularly useful volume integral. We emphasise that it is p_r that is guaranteed to be associated with ANEC violations, whereas inequalities associated with p_t generically represent "normal" matter.

Specialization 1: spatial Schwarzschild.—Now consider the special case where the spatial metric is exactly Schwarzschild; that is $b(r) \to 2m = r_0$. Then $\rho = 0$ throughout the spacetime and we get the very simple result

$$\oint p_r dV = -\int_{r_0}^\infty \ln\left[\frac{\exp[2\phi]}{1 - 2m/r}\right]dr. \tag{15}$$

Thus the total ANEC-violating component of the stress energy is finite and bounded. Suppose, in particular, that we have a wormhole whose field only deviates from Schwarzschild in the region from the throat out to radius a. Then we can further simplify the above to

$$\oint p_r dV = -\int_{r_0}^{a} \ln\left[\frac{\exp[2\phi]}{1 - 2m/r}\right] dr. \qquad (16)$$

Under this same restriction the ANEC integral satisfies

$$I < -\frac{2}{4\pi} \int_{a}^{\infty} \frac{1}{r^2} dr = -\frac{1}{2\pi a}, \qquad (17)$$

and so is strictly bounded away from zero. Now

$$\int_{r_0}^{a} \ln\left[\frac{\exp[2\phi]}{1 - 2m/r}\right] dr < \int_{r_0}^{a} \ln\left[\frac{\exp[2\phi_{max}]}{1 - 2m/r}\right] dr. \qquad (18)$$

Evaluating this last integral

$$\oint p_r dV > -(a - 2m)\ln\left[\frac{\exp[2\phi_{max}]}{1 - 2m/a}\right] - 2m\ln\left(\frac{a}{2m}\right). \qquad (19)$$

This is useful because it is an explicit *lower* bound on the total amount of radial stress in terms of ϕ_{max} and the size of the region of ANEC-violating matter. Similarly

$$\oint p_r dV < -(a - 2m)\ln\left[\frac{\exp[2\phi_{min}]}{1 - 2m/a}\right] - 2m\ln\left(\frac{a}{2m}\right). \qquad (20)$$

This is now an *upper* bound in terms of ϕ_{min} and the size of the region of ANEC-violating matter. If we now choose geometries such that ϕ_{max} and ϕ_{min} are not excessively divergent [no worse than $(a - 2m)^{-\delta}$ with $\delta < 1$], we can take the limit $a \rightarrow 2m$ to obtain

$$\oint p_r dV \rightarrow 0. \qquad (21)$$

That is, by considering a sequence of traversable wormholes with suitably chosen a and $\phi(r)$ [and $b(r) = 2m$] we can construct traversable wormholes with arbitrarily small quantities of ANEC-violating matter. (With the ANEC line integral nevertheless remaining finite and negative.) Since this result is rather important, we now provide an even more explicit example.

Specialization 2: piecewise $R = 0$ wormhole.—We now consider a segment of $R = 0$ wormhole (zero Ricci scalar) [15] truncated and embedded in a Schwarzschild geometry. For $r \in (r_0 = 2m, a)$ take

$$\exp[\phi(r)] = \epsilon + \lambda\sqrt{1 - 2m/r}, \qquad (22)$$

and for $r \in (a, \infty)$ take

$$\exp[\phi(r)] = \sqrt{1 - 2m/r}. \qquad (23)$$

Continuity implies

$$\epsilon + \lambda\sqrt{1 - 2m/a} = \sqrt{1 - 2m/a}, \qquad (24)$$

so that

$$\lambda = 1 - \frac{\epsilon}{\sqrt{1 - 2m/a}}. \qquad (25)$$

There is ANEC-violating matter confined to the region $[r_0, a)$ and a thin shell of *quasinormal* matter at $r = a$ [16]. (Quasinormal meaning it is not ANEC violating, or for that matter violating other energy conditions, namely, the weak energy condition or the strong energy condition. It does however violate the dominant energy condition; see [4] for details on the different energy conditions.)

Computing the extrinsic curvature at $r = a > 2m$:

$$K_{ab} = \frac{\partial h_{ab}}{\partial \eta} = \frac{\partial h_{ab}}{\partial r}\frac{\partial r}{\partial \eta} = \frac{\partial h_{ab}}{\partial r}\sqrt{1 - 2m/a}. \qquad (26)$$

Then the only nontrivial component is

$$K_{tt}^{+} = \sqrt{1 - 2m/a}\,\frac{2m}{a^2}, \qquad (27)$$

while

$$K_{tt}^{-} = \lambda\sqrt{1 - 2m/a}\,\frac{2m}{a^2}. \qquad (28)$$

The only nonzero component of the discontinuity is

$$[K_{tt}] = (1 - \lambda)\frac{2m}{a^2}\sqrt{1 - 2m/a} = \epsilon\frac{2m}{a^2}. \qquad (29)$$

In an orthonormal frame

$$[K_{\hat{t}\hat{t}}] = \frac{\epsilon 2m/a^2}{1 - 2m/a}. \qquad (30)$$

Since the extrinsic curvature is nonzero, the junction condition formalism implies that the metric is C^1 at $r = a$. The only component of stress energy that picks up a delta-function contribution is p_t. That is

$$p_r = -\frac{1}{8\pi}\frac{2m\epsilon}{r^3(\epsilon + \lambda\sqrt{1 - 2m/r})}\Theta(a - r); \qquad (31)$$

$$p_t = \frac{1}{8\pi}\frac{m\epsilon}{r^3(\epsilon + \lambda\sqrt{1 - 2m/r})}\Theta(a - r) \\ + \frac{1}{8\pi}\frac{\epsilon 2m/a^2}{1 - 2m/a}\delta(\eta - \eta_a). \qquad (32)$$

Note that at the throat ($r = 2m$)

$$p_r = -\frac{1}{8\pi}\frac{1}{(2m)^2} = -\frac{p_t}{2}, \qquad (33)$$

both of which are finite for all a and ϵ.

The volume integral is

$$\oint p_r dV = -2m\epsilon \int_{2m}^{a} \frac{dr}{r(\epsilon + \lambda\sqrt{1 - 2m/r})}, \qquad (34)$$

which evaluates (after a little work) to

$$\oint p_r dV = -2m\epsilon\epsilon_s \left\{ \frac{\epsilon\epsilon_s \ln[\epsilon^2(1-\epsilon_s^2)/\epsilon_s^2] + (\epsilon - \epsilon_s)\ln[(1-\epsilon_s)/(1+\epsilon_s)]}{(\epsilon-\epsilon_s)^2 - \epsilon^2\epsilon_s^2} \right\}. \tag{35}$$

Here we have introduced

$$\epsilon_s = \sqrt{1 - 2m/a}. \tag{36}$$

Once $\oint p_r dV$ is known, $\oint p_t dV$ is trivial. It is easiest to consider

$$\oint [p_r + 2p_t]dV = \frac{2\epsilon m}{\sqrt{1 - 2m/a}} = 2\frac{\epsilon}{\epsilon_s}m. \tag{37}$$

The point is that $\oint p_r dV$ can be made arbitrarily small by suitably choosing ϵ and a. For example, take ϵ/ϵ_s fixed, and let $\epsilon \to 0$. Now in that case $\oint [p_r + 2p_t]dV$ remains finite, but you could just as easily choose $\epsilon_s = \sqrt{\epsilon} \to 0$, or even ϵ_s fixed and $\epsilon \to 0$, in which case both integrals tend to zero. Thus the volume integrals of both p_r and p_t can be made arbitrarily small while the volume integral of ρ is identically zero by construction. In this particular example the geometry is sufficiently simple that an integration in terms of the proper volume $\sqrt{g_3}d^3x$ can also be explicitly carried out. Qualitatively similar results are obtained.

We emphasize that the particular details of the geometry we have written down are no where near as important as the general principle that energy condition violations can be made arbitrarily small.

Conclusions.—The specific examples presented in this Letter make essential use of spherical symmetry, and it is not yet clear to us how to usefully extend these ideas to more general situations—that is however not critical to the central point of this Letter; any explicit example of a traversable wormhole with infinitesimal ANEC violations will serve to illustrate the point we wish to make.

Let us now summarize the key result: ANEC violations are certainly needed to support traversable wormholes, but by appropriate choice of the wormhole geometry the total quantity of ANEC-violating matter can be made infinitesimally small. Quantum physics is known to lead to small violations of the ANEC, and indeed quantum-induced ANEC violations are known to be an essential precondition for the violation of the area-increase theorem engendered by the existence of Hawking radiation. Thus topological censorship shares with the area-increase theorem the fact that its conclusions can be radically altered by subtle quantum effects. (In contrast, generating macroscopic violations of the positive mass theorem requires macroscopic violations of the energy conditions.)

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[1] M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988); M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).

[2] J. L. Friedman, K. Schleich, and D. M. Witt, Phys. Rev. Lett. **71**, 1486 (1993); **75**, 1872(E) (1995); K. Schleich and D. M. Witt, gr-qc/9903061; G. J. Galloway, K. Schleich, D. M. Witt, and E. Woolgar, Phys. Rev. D **60**, 104039 (1999).

[3] D. Hochberg and M. Visser, Phys. Rev. Lett. **81**, 746 (1998); Phys. Rev. D **56**, 4745 (1997).

[4] M. Visser, *Lorentzian Wormholes: from Einstein to Hawking* (AIP Press, Woodbury, New York, 1995).

[5] H. Epstein, V. Glaser, and A. Jaffe, Nuovo Cimento **36**, 1016 (1965).

[6] D. Hochberg, A. Popov, and S. V. Sushkov, Phys. Rev. Lett. **78**, 2050 (1997).

[7] L. H. Ford and T. A. Roman, Phys. Rev. D **51**, 4277 (1995); A. Borde, L. H. Ford, and T. A. Roman, Phys. Rev. D **65**, 084002 (2002); L. H. Ford, A. D. Helfer, and T. A. Roman, Phys. Rev. D **66**, 124012 (2002).

[8] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).

[9] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982), p. 270.

[10] R. Schon and S. T. Yau, Commun. Math. Phys. **65**, 45 (1979); **79**, 231 (1981).

[11] C. Barcelo and M. Visser, Int. J. Mod. Phys. D **11**, 1553 (2002).

[12] W. Israel, Nuovo Cimento B **44**, 1 (1966); **48**, 463(E) (1967); K. Lanczos, Ann. Phys. (Leipzig) **74**, 518–540 (1924); N. Sen, Ann. Phys. (Leipzig) **73**, 365–396 (1924).

[13] In order to be certain that m_∞ corresponds to the physical mass, there is a technical requirement on the asymptotic behavior of $g_{tt}g_{rr} \to -1 + O(1/r^2)$.

[14] If one is dealing with a star instead of a wormhole one obtains a similar formula with the right-hand side of (13) being modified by an (i) overall factor of one-half and (ii) the integration limits becoming 0 to ∞.

[15] N. Dadhich, S. Kar, S. Mukherji, and M. Visser, Phys. Rev. D **65**, 064004 (2002).

[16] The thin shell is present only as a mathematical convenience—any smooth transition from the wormhole region to the Schwarzschild region would be physically acceptable, but explicit evaluation of the integrals would be more difficult.