Ray-Splitting Correction to the Weyl Formula: Experiment versus Theory

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An abrupt change in the dielectric constant of media filling a microwave cavity was predicted by Prange *et al.* [Phys. Rev. E **53**, 207 (1996)] to produce a *ray splitting* correction to the Weyl formula for the mean staircase function of resonances. We present the first experimental confirmation of this effect. Our results with a quasi-two-dimensional cavity are directly relevant to the ray-splitting correction in two-dimensional quantal ray-splitting billiards.

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The number $N(f)$ of resonances that a resonator has below frequency f is an essential concept in physics. It may be written as $N(f) = \Sigma_j \theta(f - f_j)$, where $\theta(x) = 0$, 1/2, 1, respectively, for $x < 0$, $x = 0$, and $x > 0$. Since $N(f)$ jumps at each resonance, $N(f)$ is called the spectral *staircase function* [1]. It, or its derivative $\rho(f) = dN(f)/df$, the density of modes, figures prominently in the derivation of, e.g., Planck's radiation law [2] and the specific heats of solids [3] and in concert hall acoustics [4]. Though fluctuations of $N(f)$ around its smooth mean, $\bar{N}(f)$, have important statistical properties [5–9], knowing $\bar{N}(f)$ suffices for many physical applications. Investigation of $\bar{N}(f)$ has a rich and growing history.

Lecturing in 1910 in Göttingen, the physicist Lorentz aroused the interest of mathematicians with a conjecture that, for large f, $\bar{N}(f)$ for a 3D electromagnetic resonator is proportional to its volume, independent of its shape [10]. Though his mathematician host Hilbert (supposedly [4,11]) predicted that the proof would not be done in his lifetime, Weyl succeeded within a year to derive the leading term of $\bar{N}(f)$ for generic resonators of physical relevance [12,13]. In $d = 3$, 2, or 1 dimension, $\bar{N}(\text{large } f)$ is proportional to f^d times, respectively, the volume, surface area, or length of the resonator [14,15].

Though Weyl conjectured (correctly as it turned out [14]) the first correction term, which has opposite signs for Dirichlet (−) and Neumann (+) boundary conditions, acoustics researchers were the first to need low-f corrections. Independently, Bolt [16] and Maa [17] theoretically examined the acoustical spectrum of a particular 3D case, viz., the undamped, closed, rectangular parallelepiped (air-filled room). Each derived a formula [18] for $\bar{N}(f)$. Maa's formula for $\bar{N}(f)$ [see Eq. (15) in [17]] has the "modern" form: a volume (Weyl) term successively corrected by a positive (Neumann) surface area term, a length term, and a constant term. Later work on generic 3D resonators [14] showed that the magnitude of the length correction depends on whether the surface is smooth or has sharp edges. Bolt [19] used quasi-2D acoustical cavities for—to our knowledge—the first experimental test of a correction to the Weyl (area) term.

The mathematical expression for $\bar{N}(f)$ has come to be called the *Weyl formula* (WF), even when various low-f corrections, e.g., surface, shape, length, or connectivity, are included. Details of quantal/classical correspondence emerge from exploring how terms in the WF for a given resonator (wave system) can be associated [20,21] with nonperiodic orbits of the classical billiard (ray system) having the same shape.

Recently, a new, universal, *ray-splitting* (RS) correction to the WF was predicted [22] for resonators having an abrupt change of properties at a RS boundary [23]. A mechanical example is a circular drum with a head made of two different materials joined along a diameter that becomes the RS boundary [24,25]. That, in general, a plane wave (ray) propagating towards and encountering a RS boundary is split into a reflected and transmitted wave (ray) motivates the name "ray splitting" [23].

This Letter reports the first experimental confirmation of the RS correction to the WF.

Our experiments use a microwave cavity shaped like a rectangular parallelepiped; see Figs. 1 and 2. Because its length L and width W far exceed its height H (along z), for $f < f_{2D} = c/2H = 3.45$ GHz the empty cavity has only transverse-magnetic modes [26] with electric field E_z pointing along and independent of z. For such a quasi-two-dimensional (Q2D) cavity the vector Helmholtz equation reduces to a scalar Schrödinger equation, with E_z playing the role of the wave function in an equivalent 2D quantal billiard [27]. The use of Q2D microwave cavities for the experimental simulation of quantum chaotic systems began in 1990 and developed rapidly [27–32]. For historical interest, however, we note that experimental visualization of acoustical wave functions ("interference systems") was accomplished before 1920 in 3D rooms [33] and by 1939 in Q2D enclosures [19].

In [34,35] a Q2D cavity (area $\simeq 277$ cm^2) loaded with one dielectric (Teflon) bar was used to verify the existence of non-Newtonian periodic orbits produced by RS. Our experiment uses a larger Q2D cavity [$LW = 7942(4)$ cm^2] loaded with two dielectric (paraffin wax [36]) bars [one bar has width $w_1 = (b - a) = 18.0(1)$ cm,

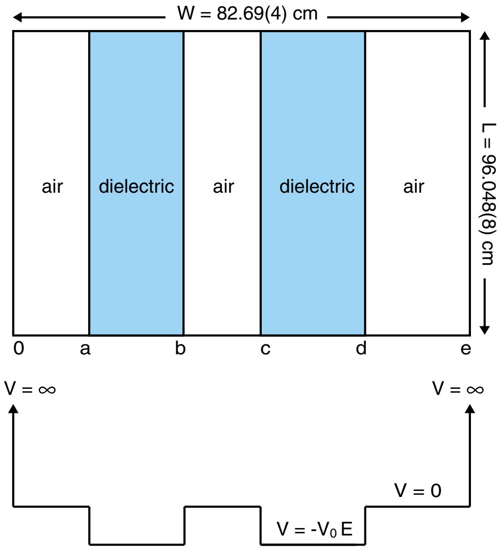

FIG. 1 (color online). Top: sketch (not to scale) defining some microwave cavity dimensions. Bottom: spatial dependence of the potential function for the equivalent, 2D quantal RS billiard.

length $l_1 = L$, height $h_1 = H$; the other bar has width $w_2 = (d - c) = 19.0(1)$ cm, length $l_2 = L$, height $h_2 = H$].

In [37] we give construction details and show that our dielectric-bar-loaded cavity is a Q2D system to at least 2.3 GHz. Therefore, this Letter presents data obtained with a system that is strictly analogous to a 2D quantal billiard with potential-step RS boundaries [38].

Briefly, aluminum bars are the cavity sidewall and copper-coated circuit boards are its top and bottom lids; clamps hold them together. The bottom lies flat on a steel table, while the wax bars and 18 carefully cut, thin plastic spacers support the top. The two paraffin bars create the air-wax dielectric RS discontinuities. Magnetic-loop antennas are inserted a few millimeters into the cavity through two of the many 3.6 mm diameter holes drilled through the sidewall. One antenna couples power $P_{\text{in}} \sim \mathcal{O}(\text{mW})$ from a Gigatronics model 1018 synthesizer into the cavity, and the other antenna couples power P_{out} out of the cavity, allowing us to obtain transmission spectra; individual resonances have a quality factor $Q \simeq (3\text{–}4) \times 10^3$. Typically, $P_{\text{out}}/P_{\text{in}} \sim \mathcal{O}(10^{-6})$, so we use two cascaded, 20 dB amplifiers to boost the output signal, which is rectified with a diode detector. Using 1 kHz square-wave amplitude modulation of the synthesizer output, we feed the diode signal to a lock-in amplifier, measure its output with a 5-1/2 digit multimeter, and record this in a computer as a function of f under general purpose interface (IEEE-488) bus control, with typical frequency step $\Delta f = 50$ kHz. We record and analyze transmission spectra for enough different placements of antennas to ensure that we neither miss valid resonances nor include spurious ones. Complete spectra include at least 152 resonances up to 1.95 GHz.

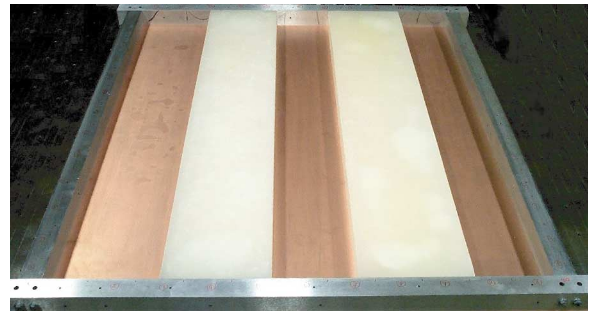

FIG. 2 (color online). Photograph of the cavity with top and antennas removed. The sidewalls have height $H = 4.341(3)$ cm. The two paraffin wax bars fill the length L and height H.

Figures 1 and 2 show the "bars-apart" configuration; there are four RS boundaries and eight RS junctions [39]. Since [39] proved that an RS junction makes no contribution to the WF for conditions met by our cavity, we need not consider RS junctions further. The top panel in Fig. 1 defines useful symbols, and the bottom panel shows the potential function for the analogous 2D quantal RS billiard. A "bars-together" configuration (not shown in the figures but discussed below) has $b = c$ and only two RS boundaries. If we keep both wax bars away from the left- and right-hand walls of the cavity, i.e., $a \neq 0$ and $d \neq e$, the WF is given by [38–40]

$$\bar{N}(E) = \frac{E}{4\pi}[A + A'(1 + V_0)] - \frac{\sqrt{E}}{4\pi}[P + P'\sqrt{1 + V_0}] + \frac{1}{4} + \bar{N}_{\text{RS}}(E) \qquad (1)$$

with $E = k^2 = (2\pi f/c)^2$. A and P (A' and P') represent the area and perimeter of the air-filled (wax-filled) part of the cavity, with dielectric constant $\kappa_e^{\text{air}} = 1.000\,58$ [$\kappa_e^{\text{wax}} = 2.236(13)$ [36]]. The RS correction is

$$\bar{N}_{\text{RS}}(E) = L_{\text{RS}}\, \nu(V_0)\sqrt{V_0 E}, \qquad (2)$$

where $V_0 = (\kappa_e^{\text{wax}} - \kappa_e^{\text{air}})/\kappa_e^{\text{air}}$, $\nu(V_0)$ is a constant for constant V_0, and L_{RS} is the total length of RS boundaries. For a bars-apart configuration, e.g., Figs. 1 and 2, $L_{\text{RS}}^{A} = 4L$; in a bars-together configuration, $L_{\text{RS}}^{T} = 2L$.

The factor-of-two difference in the RS corrections for bars-apart versus bars-together configurations is the key to extracting the RS correction in our experiment.

Measuring a complete bars-apart (bars-together) spectrum gives the staircase function $N_A(f)$ [$N_T(f)$]. According to Eq. (1), the difference $D(f) = N_A(f) - N_T(f)$ leaves the RS correction for an air-wax RS boundary of length $2L$. This way of extracting the RS correction does not depend explicitly on measurements of the cavity parameters: we *do not* need precise empty-cavity and wax-bar dimensions and dielectric constants, but we *do* need to keep them the same for bars-apart and bars-together measurements.

We implement this scheme for Q2D spectra up to 1.95 GHz as follows. We measure a complete bars-apart spectrum $A1$, with $a = 17.0$ cm, $b = 35.0$ cm, $c = 48.3$ cm, and $d = 67.3$ cm to obtain $N_{A1}(f)$, and then a complete bars-together spectrum $T1$, with $a = 15.7$ cm, $b = c = 33.7$ cm, $d = 52.7$ cm to obtain $N_{T1}(f)$. Because the RS correction, less than one state, is obscured by (reproducible) fluctuations, we integrate $D(f)$ to obtain $I(f) = \int_0^f D(f')\,df'$. Though non-negligible fluctuations still survive, this exposes the RS correction.

Averaging $I(f)$ over different bars-apart and bars-together configurations should help to reduce fluctuations further. Therefore, we measure a complete spectrum $A2$, with $a = 15.0$ cm, $b = 33.0$ cm, $c = 44.5$ cm, and $d = 63.5$ cm to obtain $N_{A2}(f)$, and $T2$ with $a = 22.4$ cm, $b = c = 40.4$ cm, and $d = 59.4$ cm to obtain $N_{T2}(f)$.

Figure 3 examines nonaveraged results. The top four panels compare the four possible, apart-together, integrated differences $A1T1$, $A1T2$, $A2T1$, and $A2T2$ to the analytical expectation $I_{ana}(f)$ obtained by integrating Eq. (2). All four panels show oscillations. The fine-scale ones [$\mathcal{O}(0.1$ GHz) spacing] are similar in all four panels, the longer-scale ones less so, but all are experimentally reproducible. Fourier transforms (shown in [37] but not here because of limited space) of the $I(f)$ curves show peaks. These can be attributed to families of periodic orbits that survive the subtraction process, thereby *emphasizing* orbits in the corresponding Q2D classical billiard that are caused by RS; see [37].

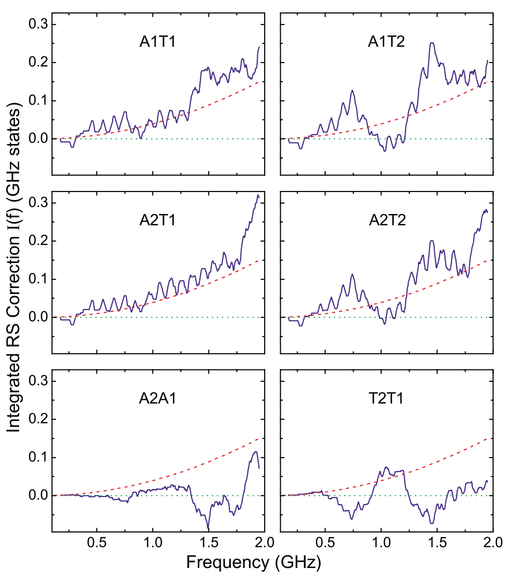

FIG. 3 (color online). Full lines: $I(f) = \int_0^f df'[N_A(f') - N_T(f')]$, integrated difference of $N(f)$ for different bars-*A*part ($A1$ and $A2$) and bars-*T*ogether configurations ($T1$ and $T2$). The top four panels are the four possible A-T differences. The bottom two panels are null tests $A2A1$ and $T2T1$. Dashed lines: analytical prediction $I_{ana}(f)$ for the integrated RS difference for the cavity in Figs. 1 and 2. Dotted lines: $I(f) = 0$ for reference.

The bottom two panels in Fig. 3, the apart-apart and together-together integrated differences, $A2A1$ and $T2T1$, are important *null tests*. Both are consistent with no RS correction, the net area under each curve being less than 5% of the area A_{ana} under the analytical curve (dashed line). Conversely, the areas under the $A1T1$, $A1T2$, $A2T1$, and $A2T2$ curves are from 1.41 to 1.46 times A_{ana}.

Figure 4 compares the average of the top four panels in Fig. 3 to the analytical curve. Averaging reduces the longer-scale fluctuations somewhat, less so the fine-scale oscillations. Above about 1.5 GHz the experimental results remain significantly above the analytical curve, but without more data, we cannot know if this is an artifact of insufficiently averaged, longer-scale fluctuations. Nevertheless, the results shown in Figs. 3 and 4 establish without a doubt the existence of the RS correction to the WF and, approximately, its f dependence.

We emphasize that our measurements provide a model-independent evaluation of the RS correction. We make no assumption about the f dependence of the RS correction nor about the magnitude of prefactors multiplying the area and perimeter terms in the WF for the dielectric-loaded cavity. Note that if only one bar or only one bar position were used, mechanical measurements of the cavity and bar and measurements of κ_e^{wax} would all have to be supremely accurate. For our cavity, fluctuations in $N(f)$ obscure the RS correction, which is about 3 orders of magnitude smaller than the area term in the WF. Furthermore, it would be difficult to separate the RS contribution from the perimeter term since both are proportional to length. Use of two wax bars and the difference in measurements for bars-apart and bars-together configurations, however, cancels out the area, perimeter, and topological contributions, leaving only the RS correction.

We carefully control the sources of three systematic effects. The first is thermal. The metal cavity and the wax bars expand thermally (at different rates), shifting

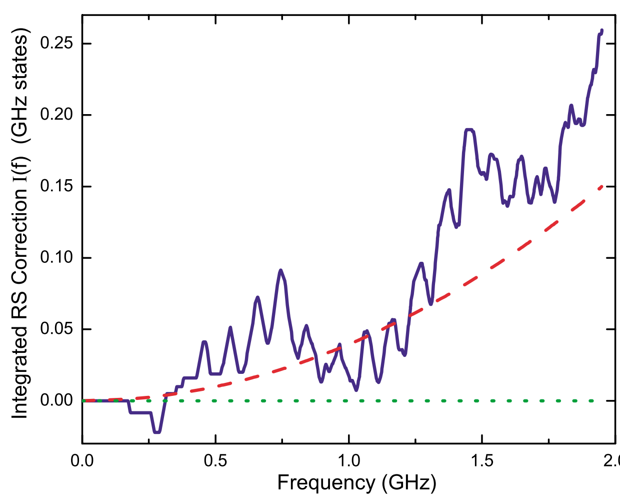

FIG. 4 (color online). Full line: average of the top four panels in Fig. 3. Dashed line: analytical prediction I_{ana} as in Fig. 3. Dotted line: $I(f) = 0$ for reference.

resonance frequencies. To compare reliably different data sets taken under ostensibly identical conditions except for the positions of antennas, we keep the temperature at 19 ± 0.5 C. The second is flatness of the top lid. In test runs we observe that (most) resonance frequencies drop when we deform the top lid to make it locally closer to the bottom lid. The likely explanation is a local increase in the capacitance C, decreasing the resonance frequency à la $\omega = 1/\sqrt{LC}$ (see [41], Sec. 23-5). The third is obtaining complete spectra. One missing or one extra resonance at frequency f^* makes an experimental $I(f)$ curve in, e.g., Figs. 3 and 4, move away from the analytical curve with a slope of ± 1(GHz state) per GHz for $f > f^*$, which is easily seen by the eye. Earlier work [34,35] that verified the existence of non-Newtonian orbits in a RS cavity also required complete spectra. Conversely, experiments using microwave cavities to address other issues in quantum chaology, e.g., level statistics, are more tolerant of missed or spurious resonances [42,43]. Level statistics alone are *not* sensitive enough to ensure the completeness of a spectrum. Supported by numerical simulations and analytical insight into the structure of the spectrum of our cavity [37], we are sure that our spectra are complete.

In conclusion, we present the first measurement of the RS correction to the Weyl formula. Our experiment using a dielectric-loaded microwave cavity gives a model-independent, quantitative result for the magnitude of this correction and a functional dependence that is reasonably close to an analytical prediction. Because our Q2D cavity is mathematically equivalent to a 2D RS billiard, our results relate directly to this quantal RS problem.

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