

Phase Transition in the 2D Random Potts Model in the Large- q Limit

J-Ch. Anglès d'Auriac¹ and F. Iglói^{2,3}

¹*Centre de Recherches sur les Très Basses Températures, B.P. 166, F-38042 Grenoble, France*

²*Research Institute for Solid State Physics and Optics, H-1525 Budapest, P.O. Box 49, Hungary*

³*Institute of Theoretical Physics, Szeged University, H-6720 Szeged, Hungary*

(Received 25 November 2002; published 13 May 2003)

Phase transition in the two-dimensional q -state Potts model with random ferromagnetic couplings is studied in the large- q limit by a combinatorial optimization algorithm and by approximate mappings. We conjecture that the critical behavior of the model is controlled by the isotropic version of the infinite randomness fixed point of the random transverse-field Ising spin chain and the critical exponents are exactly given by $\beta = (3 - \sqrt{5})/4$, $\beta_s = 1/2$, and $\nu = 1$. The specific heat has a logarithmic singularity, but at the transition point there are very strong sample-to-sample fluctuations. Discretized randomness results in discontinuities in the internal energy.

DOI: 10.1103/PhysRevLett.90.190601

PACS numbers: 05.50.+q, 64.60.Fr, 75.10.Nr

Phase transitions in the presence of quenched disorder in systems with short range interactions are usually controlled by conventional random fixed points, in which the strength of disorder remains finite under renormalization. Exact information is scarce about these problems which has greatly hampered our understanding about disordered systems.

Exact results are practically restricted to models with strictly correlated randomness, such as the McCoy-Wu model [1] in two dimensions (2D) and to random quantum spin chains, for which disorder is correlated in the (imaginary) time direction. As known from renormalization group (RG) and numerical studies about the random transverse-field Ising model (RTIM) [2] and other problems, the critical behavior of these random systems are often controlled by infinite randomness fixed points (IRFPs) [3], in which—under renormalization—the strength of disorder grows without limits. In IRFPs scaling is strongly anisotropic; thus space and time play inequivalent roles. In spite of this, some aspects of conformal invariance [4] seem to apply; cf. the scaling form of operator profiles in the RTIM are found to follow conformal predictions [5]. This observation makes it likely that the RTIM is the Hamiltonian version [6] of a 2D classical, isotropic random system, which obeys conformal symmetry and the critical exponents of which are related to the exponents of the RTIM, and thus these are presumably exactly known.

In this Letter we conjecture that the possible candidate of this isotropic system is the q -state Potts model [7] with random ferromagnetic bonds [random bond Potts model (RBPM)] in the large- q limit. In this limit the high-temperature expansion of the model is dominated by a single diagram [8], which is determined by a very efficient combinatorial optimization algorithm [9]. The calculated bulk and surface magnetization scaling dimensions, x and x_s , respectively, are found in good agreement with the corresponding exact values for the RTIM [2]:

$$x = (3 - \sqrt{5})/4, \quad x_s = 1/2. \quad (1)$$

The conjectured isomorphism between the two problems is further supported by topological similarities in the two structures and by a calculation performed in the strongly anisotropic limit of the RBPM.

The RBPM is defined by the Hamiltonian

$$-H/kT = \sum_{\langle ij \rangle} K_{ij} \delta(\sigma_i, \sigma_j), \quad (2)$$

where the spin variable at site i is $\sigma_i = 0, 1, \dots, q-1$, $K_{ij} > 0$ are random ferromagnetic couplings, and the summation runs over nearest neighbor pairs. In the random cluster representation the partition function of the system is expressed in terms of $v_{ij} = \exp K_{ij} - 1$ as [8]

$$Z = \sum_F q^{C(F)} \prod_{ij \in F} v_{ij}, \quad (3)$$

where the summation runs over all subsets of bonds, F , and $C(F)$ is the number of connected components of F , counting also the isolated sites. Having the parametrization, $v_{ij} = q^{\alpha_{ij}}$, the partition function is expressed as

$$Z = \sum_F q^{f(F)}, \quad f(F) = C(F) + \sum_{ij \in F} \alpha_{ij}, \quad (4)$$

which in the large- q limit is indeed dominated by the largest contribution, $f^* = \max_F f(F)$, and the partition function is asymptotically given by $Z = Nq^{f^*}$ where the number of optimal sets (OSS), N , is likely to be one.

According to rigorous results [10] the internal energy of the 2D RBPM is continuous at the phase-transition point for any q , if the probability distribution of the couplings is absolutely continuous. In numerical calculations [11,12] performed at the phase-transition point, which is known by duality [13], universal, i.e., disorder independent critical, behavior has been observed even for atomistic (cf. bimodal) distributions. The scaling dimension, x , is a monotonously increasing function of q , but its saturation value in the large- q limit is difficult to be

estimated due to strong logarithmic corrections [14]. The correlation length exponent is close to $\nu = 1$, for any value of q . Note, however, that by transfer matrix calculation Jacobsen and Cardy [12] have obtained $\nu < 1$ for the bimodal distribution of disorder, which violates the rigorous upper bound, $\nu \geq 2/d$ [15].

In the large- q limit the critical properties of the RBPM are related to the structure of the clusters in the OS in close analogy with percolation [16]. In the paramagnetic phase there are only finite clusters in the OS and the linear extent of the largest clusters is used to define the correlation length, ξ . In the ferromagnetic phase there is an infinite cluster and the ratio of lattice points belonging to it is related to the (finite) magnetization of the RBPM. Finally, at the phase-transition point the largest (infinite) cluster is a fractal, and its fractal dimension, d_f , is related to the magnetization scaling dimension as $d = d_f + x$. Similarly, the sum of the surface fractal dimension of the percolating cluster, d_f^s , and the anomalous dimension of the surface magnetization, x_s , gives the Euclidean dimension of the surface: $d^s = 1 = d_f^s + x_s$.

In Ref. [8] the OS was approximately calculated by the simulated annealing method on relatively small lattices (up to 24×24), and the critical exponent $x = 0.17-0.19$ was consistent with the conjectured value in Eq. (1). In this Letter we apply a recently developed combinatorial optimization algorithm [9] with which we can determine the *exact* OS in strongly polynomial time. With this algorithm we could treat far larger systems as before, and averages and distributions were calculated for systems with $L = 32, 64, 128$, and 256 over at least a thousand disorder realizations. As a consequence, our estimates of the critical exponents become much more accurate than before, and we also studied, at the first time, the surface properties of the RBPM.

In the calculations the couplings and thus the parameters, α_{ij} , were taken from distribution in which the critical point is exactly known from self-duality, when $P(\alpha - 1/2) = P(1/2 - \alpha)$ [13]. Mostly we used the bimodal distribution: $P_b(\alpha) = [\delta(w + \Delta w - \alpha) + \delta(w - \Delta w - \alpha)]/2$ with $w > \Delta w > 0$, in which the distance of the critical temperature, t , is measured by $t \approx 1 - 2w$. Having the parameter $\Delta w = 1/3$ the microscopic length scale in the problem [8], $l_c \approx (2\Delta w)^{-2}$, was not too large. We also used the continuous (uniform) distribution $P_u(\alpha) = 1/u$, for $0 \leq \alpha < u$ and zero otherwise, in which the distance from the critical point is given by $t \approx 1 - u$. To calculate bulk (surface) quantities we applied periodic (open) boundary conditions (BCs).

First, we considered the behavior of the system at the transition point, when the cluster structure of a typical OS is shown in Fig. 1. As illustrated in the middle of Fig. 1 the OS is self-similar and its topology, being isotropic, can be conveniently represented by the connectivity structure (CS) of the OS at a given line, as shown in the bottom of Fig. 1.

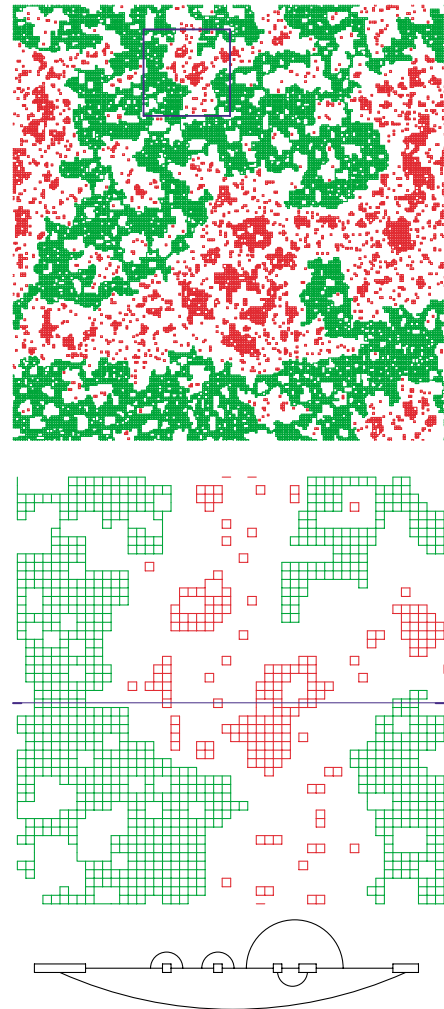


FIG. 1 (color online). Top: OS for a typical disorder realization at the critical point on a 256×256 lattice with periodic boundary conditions. Percolating and finite clusters are marked with gray and dark lines, respectively. Middle: Enlargement of a square proportion of size 48×48 in the upper-middle part of the OS to illustrate self-similarity. Bottom: The connectivity structure of the OS along a line, which consists of six connected units (“spins”) and five open units (“bonds”).

We have checked that the largest connected cluster is indeed a fractal and its fractal dimension is calculated from an analysis of the probability distribution function, $R(m, L)$, which measures the fraction of clusters having a size at least m . According to scaling theory [16], $R(m, L)$ asymptotically behaves as

$$R(m, L) = m^{-\tau} \tilde{R}(m/L^{d_f}), \quad (5)$$

with $\tau = (2 - d_f)/d_f$. With the conjectured value of $d_f = 2 - x = (5 + \sqrt{5})/4 = 1.809$ we obtained a very good scaling collapse, which is shown in Fig. 2. To characterize the accuracy of the collapse we have shown in inset (a) of Fig. 2 the surface of the overlap of the scaled curves with varying values of d_f . The best collapse

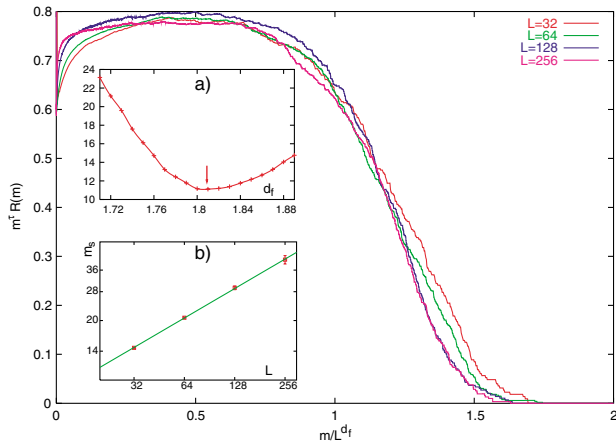


FIG. 2 (color online). Scaling collapse of the reduced probability distribution function, $R(m, L)m^{\tau}$, in Eq. (5) with $d_f = (5 + \sqrt{5})/4 = 1.809$. Inset (a): Surface of the collapse region for different values of d_f . The arrow shows the conjectured value. Inset (b): Average mass of surface sites of the largest cluster vs linear size of the system in a log-log plot. The slope of the straight line corresponds to $d_f^s = 0.495(10)$.

is indeed around the conjectured value, so that our estimate is $x = 0.190(5)$.

Next, we considered the surface magnetization properties of the model by using open BCs and measuring the average mass of surface sites of the largest cluster, m_s . As shown in inset (b) of Fig. 2, it has an asymptotic size dependence $m_s \sim L^{d_f^s}$, with $d_f^s = 0.495(10)$, which is in very good agreement with the conjectured value of $d_f^s = 1 - x_s = 1/2$; see Eq. (1).

In the following we analyze in more details the fractal structure of the OS and point out the topological similarities with the ground state wave function of the RTIM, which could explain the appearance of the same critical exponents in the two problems. As already noted in the bottom of Fig. 1 the CS consists of connected units (CUs) (corresponding to spins in the RTIM) of variable length, l_s , and moments, μ , and of open units (OUs) (corresponding to bonds in the RTIM) of variable length, l_b . If two neighboring CUs, with parameters l_s^1, μ^1 , and l_s^2, μ^2 , separated with an OU of l_b , belong to the same cluster, it is merged to an effective CU (represented by a connecting line in Fig. 1), with length $l_s^{12} = l_s^1 + l_s^2 + l_b$ and moment $\mu^{12} = \mu^1 + \mu^2$. This is precisely equivalent to a strong bond decimation in the RTIM, which is one ingredient of the strong disorder renormalization group (SDRG) method [2,17]. Similarly, if a CU with l_s, μ , and neighboring bonds of lengths l_b^1 and l_b^2 is isolated, it does not contribute to any larger cluster; therefore — at larger length scales — it can be eliminated (represented by an overgoing line in Fig. 1) and the new effective bond has a length of $l_b^{12} = l_b^1 + l_b^2 + l_s$. This process is equivalent to a strong field decimation in the SDRG for the RTIM. Thus we can conclude that for any CS of the OS in the RBPM one can construct an equivalent ground state of the

RTIM, and one can give a set of couplings, J_i , and transverse fields, h_i , for which the given ground state is realized. We recall that in the RTIM the bulk (surface) magnetization is related to the average moment of a bulk (surface) effective spin [2] in close analogy with the computation of the same quantities in the OS of the RBPM. Now keeping in mind that in both problems the topology of states is dominated by disorder effects, it is natural to assume that the statistics of the appearance of states with equivalent topology is asymptotically similar, which leads to the conjectured relation about the critical exponents in Eq. (1).

To study the energy density of the model outside the transition point, we have determined the OS at different temperatures. In a finite system of size, L , however, there are only a finite number of different OSs, their typical number being L , independently of the type of disorder. [Two neighboring OSs differ in average by one line ($\sim L$) of edges.] For a given sample the free energy is a piecewise linear function of t and the internal energy is a steplike function having typically L steps. Averaging over disorder the average internal energy becomes continuous for continuous distributions (see Fig. 3), whereas for discrete distributions some discontinuities, located at special isolated points, remain. There is a discontinuity at the phase-transition (self-dual) point, as we illustrate with the sequence of finite-size latent heats: $[\Delta E(32)]_{\text{av}} = 0.0456(90)$, $[\Delta E(64)]_{\text{av}} = 0.0477(52)$, $[\Delta E(128)]_{\text{av}} = 0.0484(28)$, and $[\Delta E(256)]_{\text{av}} = 0.0474(14)$ for the bimodal distribution with $\Delta w = 1/3$, which approach a finite value in the thermodynamic limit. The discontinuities in the internal energy are due to degeneracies, which are connected also to finite clusters. The contribution of the largest cluster only, which has a diverging size ξ , is

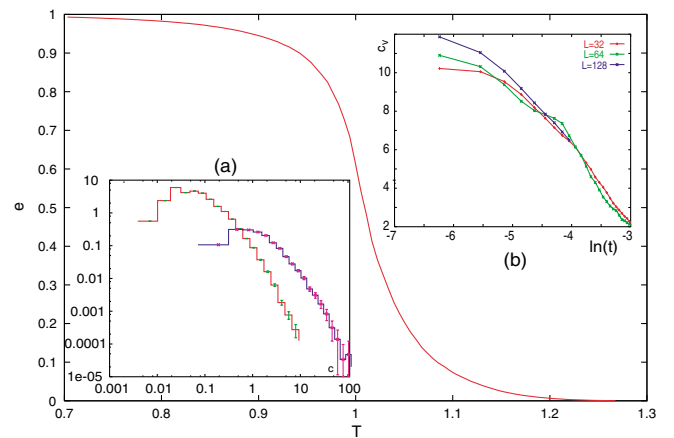


FIG. 3 (color online). Average internal energy for the continuous distribution. Inset (a): Distribution of the finite-size latent heat at the right side of the transition point for the bimodal distribution for $L = 32$ (left) and $L = 128$ (right). Inset (b): The average specific heat for the continuous distribution as a function of $\ln|t|$ in the region $Lt \gg 1$.

expected to cause singularity in the specific heat at the two sides of the transition point.

This true singularity of the specific heat, which is expected to be independent of the type of disorder, is investigated numerically, in which case, as usual, the thermodynamic limit was taken first and then approached the transition point. Close to the transition point, in the order of $|t| \sim 1/L$, we observed large sample to sample fluctuations. The distribution of the finite-size specific heat, C_v^l , defined for a given sample as the ratio of the distance between the first two energy steps and the corresponding temperature difference, as shown in inset (a) of Fig. 3 for the bimodal distribution, has a broad, power-law tail and a diverging second moment. This fact illustrates the strong randomness character of the transition, which remains qualitatively the same even for a continuous distribution of disorder. Therefore the thermodynamic singularity of the specific heat is analyzed for $|t| > 1/L$, which according to inset (b) of Fig. 3 has a logarithmic singularity of the form of $C_v(t) \sim (\ln|t|)^\epsilon$, with $\epsilon \leq 1$. Thus the specific heat exponent is $\alpha = 0$ and the correlation length exponent is $\nu = 1$. This latter result is half of the similar exponent of the RTIM [2], which means that for both problems the lower bound $\nu = 2/d$ [15] is saturated.

To close our Letter we present a consideration in the anisotropic version of the RBPM, which gives further support for the conjectured isomorphism. Here we take different couplings in the two directions ($\kappa = 1, 2$), and the bimodal distribution is generalized with $\alpha \rightarrow \alpha^{(\kappa)}$, $w \rightarrow w^{(\kappa)}$, $0 < \Delta w < w^{(1)} \leq w^{(2)}$, and $l_c \rightarrow l_c^{(\kappa)} \approx (w^{(\kappa)} / \Delta w)^2$. We choose a strongly anisotropic limit, $l_c^{(1)} = O(1)$, and $l_c^{(2)} = l_c^{(1)}L$, when the OS of the $l_c^{(2)} \times l_c^{(2)}$ system is effectively composed of $k = 1, 2, \dots, L$ almost homogeneous (“empty” or “occupied”) striplike units. For each a cost function is defined: $\epsilon(k) = \sum_{ij \in k} \alpha_{ij} - l_c^{(1)}l_c^{(2)} = O(1)$, which are Gaussian random variables.

As an illustration of the method we analyze the surface magnetization at the transition point, $w^{(1)} + w^{(2)} = 1$. Here fixed-free BC is applied; i.e., the L th unit is occupied and the magnetization at the surface, i.e., in the first unit, is of $O(1)$ if the connected cluster extends from L to 1. It is easy to see that the first unit is occupied, if and only if the second unit is occupied and $\epsilon(1) > 0$. Similarly, the necessary and sufficient condition of the occupation of the second unit is (i) the occupation of the third unit and (ii) $\epsilon(2) + \epsilon(1) > 0$. Repeating this reasoning, we obtain if $\sum_{k=1}^l \epsilon(k) > 0$ for $l = 1, 2, \dots, L$, then all units are occupied; thus $m_s(L) = O(1)$, otherwise $m_s(L) = 0$. The average surface magnetization is related to the fraction of disorder realizations with $m_s(L) = O(1)$, which can be calculated through a random walk (RW) mapping (see the analogous construction for the RTIM in Ref. [5]) as the fraction of surviving RWs of length L . Thus $[m_s(L)]_{av} \sim L^{-1/2}$ and $x_s = 1/2$, as announced in Eq. (1). A similar analysis of the bulk magnetization

along the lines of the corresponding RW mapping in Ref. [18] leads to the value of x in Eq. (1).

To conclude, our investigations have shown that the critical behavior of the RBPM in the large- q limit is dominated by strong disorder effects and possibly related to the IRFP of the RTIM. Our investigations can inspire related studies giving another insight to the conjectured results. Implementation of an RG treatment—in the same spirit as the SDRG for the RTIM—can probably be performed in the 1D cut geometry. Another promising possibility is to construct a novel (conformal) field theory for strongly random systems. Finally, our results can be used for an $1/\ln q$ expansion for large finite qs .

We are indebted to R. Juhász, M. Preissmann, H. Rieger, A. Sebő, and L. Turban for stimulating discussions. This work has been supported by the Hungarian National Research Fund under Grants No. OTKA TO34183, No. TO37323, No. MO28418, and No. M36803, by the Ministry of Education under Grant No. FKFP 87/2001, and by the EC Centre of Excellence (No. ICA1-CT-2000-70029).

-
- [1] B. M. McCoy and T. T. Wu, Phys. Rev. **176**, 631 (1968); **188**, 982 (1969); B. M. McCoy, Phys. Rev. **188**, 1014 (1969).
 - [2] D. S. Fisher, Phys. Rev. Lett. **69**, 534 (1992); Phys. Rev. B **51**, 6411 (1995).
 - [3] O. Motrunich *et al.*, Phys. Rev. B **61**, 1160 (2000).
 - [4] For a review see J. L. Cardy, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 1987), Vol. 11.
 - [5] F. Iglói and H. Rieger, Phys. Rev. B **57**, 11404 (1998).
 - [6] J. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
 - [7] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).
 - [8] R. Juhász, H. Rieger, and F. Iglói, Phys. Rev. E **64**, 056122 (2001).
 - [9] J.-Ch. Anglès d’Auriac *et al.*, J. Phys. A **35**, 6973 (2002).
 - [10] M. Aizenman and J. Wehr, Phys. Rev. Lett. **62**, 2503 (1989); **64**, 1311(E) (1990).
 - [11] M. Picco, Phys. Rev. Lett. **79**, 2998 (1997); C. Chatelain and B. Berche, Phys. Rev. Lett. **80**, 1670 (1998); Phys. Rev. E **58**, R6899 (1998); **60**, 3853 (1999); T. Olson and A. P. Young, Phys. Rev. B **60**, 3428 (1999).
 - [12] J. Cardy and J. L. Jacobsen, Phys. Rev. Lett. **79**, 4063 (1997); J. L. Jacobsen and J. L. Cardy, Nucl. Phys. **B515**, 701 (1998).
 - [13] W. Kinzel and E. Domany, Phys. Rev. B **23**, 3421 (1981).
 - [14] J. L. Jacobsen and M. Picco, Phys. Rev. E **61**, R13 (2000); M. Picco (unpublished).
 - [15] J. T. Chayes *et al.*, Phys. Rev. Lett. **57**, 2999 (1986).
 - [16] For a review see D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor and Francis, London, 1992).
 - [17] S. K. Ma, C. Dasgupta, and C.-K. Hu, Phys. Rev. Lett. **43**, 1434 (1979); C. Dasgupta and S. K. Ma, Phys. Rev. B **22**, 1305 (1980).
 - [18] H. Rieger and F. Iglói, Europhys. Lett. **45**, 673 (1999).