Analytic Solutions to the Vlasov Equations for Expanding Plasmas

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The renormalization-group approach is applied to derive an exact solution to the self-consistent Vlasov kinetic equations for plasma particles in the quasineutral approximation. The solutions obtained describe three-dimensional adiabatic expansion of a plasma bunch with arbitrary initial velocity distributions of the electrons and ions. The solution found is illustrated by the examples on ion acceleration in a plasma with hot electrons and in a plasma with light impurity ions.

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Theoretically plasma expansion into a vacuum has been studied for almost 40 years, since the work by Gurevich *et al.* [1]. However, until the last decade this problem has been treated only by using hydrodynamic models [2]. In the 1990s the kinetic aspects of plasma expansion already prevailed. The latest developments in the kinetic theory of plasma expansion aim to better understand ion acceleration in an expanding collisionless plasma bunch heated by an ultrashort laser pulse. Kinetic treatment of this problem is based on a solution to the Cauchy problem for the coupled Vlasov equations for electrons and ions in a self-consistent electric field. We emphasize that the physics of such an expansion differs from the physics of a semi-infinite collisionless plasma expansion [3–5] where the isothermal regime is enabled because of an infinite source of plasma energy. In an expanding plasma bunch, both the electrons and the ions cool adiabatically as their thermal energy transfers to the kinetic energy of the plasma expansion.

Simulations [6,7] have demonstrated the regime of plasma expansion which is characterized by electrons cooling in time. Moreover, these simulations have proven the existence of self-similar solutions to the kinetic equations. Later, a number of self-similar kinetic solutions have been found analytically [8–10]. An important step in the analytical three-dimensional kinetic theory of plasma bunch expansion with adiabatic cooling of the particles has been taken in Ref. [11]. An exact self-similar solution to Vlasov equations [11] describes the particular case with quadratic dependence of the electrostatic potential on the coordinates. Correspondingly, the initial conditions imply the same dependence of the distribution functions (DFs) on the coordinates and velocities. A more general class of kinetic solutions was found in the one-dimensional case in Ref. [10]. It was derived for arbitrary initial DFs by applying the renormalization-group approach, in which the solution to the initial-value problem is found perturbatively for $t \to 0$ and is continued in time using the renormalization-group symmetries (RGS). The theory [10] gives in analytical form the DFs for electrons and ions in the expanding plasma foil for initial conditions of practical interest for laser-plasma interaction. This includes a two-temperature Maxwellian electron DF, a super-Gaussian electron DF, and a plasma with ion multispecies. Clearly, the great advantage would be the generalization of this theory to the three-dimensional case. In this Letter we present such a theory.

Progress in the application of the RGS method [12] makes it now possible to routinely use it for different problems in laser-plasma interactions [10,13,14]. We apply this method to the Cauchy problem for the Vlasov equations

$$\partial_t f^\alpha + (\boldsymbol{v}\boldsymbol{\nabla}) f^\alpha - (e_\alpha/m_\alpha)(\boldsymbol{\nabla}\Phi)\nabla_{\boldsymbol{v}} f^\alpha = 0, \qquad f^\alpha|_{t=0} = f_0^\alpha(\boldsymbol{r}, \boldsymbol{v}). \tag{1}$$

Particle DFs $f_\alpha(t, \boldsymbol{r}, \boldsymbol{v})$ for the electrons ($\alpha = e$) and ions $\alpha = 1, 2, \ldots$) are assumed to satisfy the quasineutrality conditions,

$$\int d\boldsymbol{v} \sum_\alpha e_\alpha f^\alpha = 0, \qquad \int d\boldsymbol{v}\boldsymbol{v} \sum_\alpha e_\alpha f^\alpha = 0, \tag{2}$$

identical to those used in Ref. [11]. The electric potential is expressed in terms of the DF moments:

$$\boldsymbol{\nabla}\Phi = -\int d\boldsymbol{v} \sum_\alpha e_\alpha \boldsymbol{v}(\boldsymbol{v}\boldsymbol{\nabla}) f^\alpha \left\{ \int d\boldsymbol{v} \sum_\alpha \frac{e_\alpha^2}{m_\alpha} f^\alpha \right\}^{-1}. \tag{3}$$

The key idea of our approach is to find the RGS providing an invariance of the solution to the initial-value problem for $t \to 0$: $f^\alpha = \mathcal{F}^\alpha(t, \boldsymbol{r}, \boldsymbol{v}) \equiv f_0^\alpha(\boldsymbol{r}, \boldsymbol{v}) + O(t)$ and to find the finite transformations which extend this solution to the solution for $t > 0$.

The RGS can be found as a subgroup of the group of point Lie transformations which is admitted by Eqs. (1) and (2). The corresponding infinitesimal operators,

$$X_0 = \partial_t, \qquad X_1 = \partial_{\boldsymbol{r}}, \qquad X_2 = t\partial_t - \boldsymbol{v}\partial_{\boldsymbol{v}},$$
$$X_3^{ik} = r^k\partial_{r^i} + v^k\partial_{v^i}, \qquad i,k = 1,2,3,$$
$$X_4^{ik} = tr^k\partial_{r^i} + (r^k + tv^k)\partial_{v^i}, \qquad i,k = 1,2,3, i \neq k,$$
$$X_5 = \sum_\alpha f^\alpha \partial_{f^\alpha}, \qquad X_6 = t\partial_{\boldsymbol{r}} + \partial_{\boldsymbol{v}}, \tag{4}$$
$$X_7 = t^2\partial_t + t\boldsymbol{r}\partial_{\boldsymbol{r}} + (\boldsymbol{r} - \boldsymbol{v}t)\partial_{\boldsymbol{v}},$$
$$X_\alpha = \frac{1}{Z_{\alpha+1}}\partial_{f^{\alpha+1}} - \frac{1}{Z_\alpha}\partial_{f^\alpha},$$

arise as a generalization of the previously obtained results for the one-dimensional case [10]. Here $Z_\alpha = e_\alpha/|e|$ and the index $\alpha + 1$ denotes the particle species which follows species α. The number of generators X_α is one less than the number of plasma components.

To find the RGS one should pick a linear combination from the set of operators (4) to provide the solution found with perturbation theory for $t \to 0$ to be invariant. This leads to specific restrictions on the form of initial DFs. We consider a particular RGS generator as the linear combination of two generators X_0 and X_7 from Eqs. (4),

$$R = (1 + \Omega^2 t^2)\partial_t + \Omega^2 t\boldsymbol{r}\partial_{\boldsymbol{r}} + \Omega^2(\boldsymbol{r} - \boldsymbol{v}t)\partial_{\boldsymbol{v}}, \tag{5}$$

with the constant Ω. This is the only operator which selects the spatially symmetric initial DFs. The value Ω can be treated as the ratio of the ion acoustic velocity to the gradient length L_0. Such a solution is the particular one from the entire class of solutions to the initial-value problem which can be found by using the generators (4). For instance, a linear combination of the generators X_0, X_7, and X_4^{ik} gives rise to asymmetry in space for the solutions. However, this paper examines the possibility of using the latest developments in the RGS theory for particle kinetics and is not intended to be comprehensive, but rather to be restricted to a discussion of some new symmetric solutions.

The operator R (5) defines transformation of the phase space $(\boldsymbol{r}', \boldsymbol{v}')$ at $t = 0$ to the phase space $(\boldsymbol{r}, \boldsymbol{v})$ at $t > 0$ as follows:

$$\mathbf{r}' = \frac{\boldsymbol{r}}{\sqrt{1 + \Omega^2 t^2}}, \qquad \boldsymbol{v}'^2 + \Omega^2\boldsymbol{r}'^2 = \boldsymbol{v}^2 + \Omega^2(\boldsymbol{r} - \boldsymbol{v}t)^2. \tag{6}$$

The DFs are invariants of the renormalization-group transformations, i.e.,

$$f^\alpha = f_0^\alpha(I^\alpha), I^\alpha = \frac{1}{2}(\boldsymbol{v}'^2 + \Omega^2\boldsymbol{r}'^2) + \frac{e_\alpha}{m_\alpha}\Phi_0(\boldsymbol{r}'). \tag{7}$$

Equations (6) and (7) explicitly define the sought-for finite group (renormalization-group) transformations so that the DFs at the given time, $t \neq 0$, can be expressed in terms of their initial values.

To find the symmetry group, which is admitted by Eqs. (1) and (2), the electric field $E(t, \boldsymbol{r})$ is considered to be an unknown function of coordinates and time. Computation of this group leads to a partial differential equation of first order for the electric field. Correspondingly, the solution for the electric potential is given by

$$\Phi(t, \boldsymbol{r}) = \Phi_0(\boldsymbol{r}')(1 + \Omega^2 t^2)^{-1}. \tag{8}$$

The dependence of Φ_0 on self-similar variable $\boldsymbol{r}' = \boldsymbol{r}/\sqrt{1 + \Omega^2 t^2}$ is defined by the quasineutrality condition (2). Equations (6) and (7) present the entire class of the spatially symmetric solutions which include the original Dorozhkina and Semenov model [11] as one particular case.

The general form of the DFs, f^α, is a function of I^α

$$I^\alpha = \frac{U^2}{2} + (1 + \Omega^2 t^2)\left(\frac{\boldsymbol{v}_\perp^2 + (v - u)^2}{2} + \frac{e_\alpha\Phi}{m_\alpha}\right), \tag{9}$$

where v is the radial component of particle velocity $\boldsymbol{v}$, $\boldsymbol{v}_\perp$ is the perpendicular one, $u = rt\Omega^2/(1 + \Omega^2 t^2)$ is the local plasma velocity along the radius, and $U = r\Omega/\sqrt{1 + \Omega^2 t^2}$. Because both the kinetic equations and the quasineutrality conditions are invariant with respect to the renormalization group of point transformations, the solutions found satisfy the quasineutrality conditions at an arbitrary time, if they satisfy them at the initial time. We further illustrate the plasma expansion with two basic examples of practical interest.

Example 1.—At the initial time $t = 0$, the ions are assumed to obey a Maxwellian DF with density n_{i0} and temperature T_i and the electrons obey a two-temperature Maxwellian DF with densities and temperatures of the cold and hot components n_{c0} and n_{h0} ($n_{c0} + n_{h0} = Zn_{i0}$) and T_e and T_h, respectively. In this case the DFs may be expressed as

$$f^e = \frac{Zn_{i0}}{(2\pi)^{3/2}v_{Te}^3}\left[(1 - \rho)e^{-I^e/v_{Te}^2} + \rho\left(\frac{T_e}{T_h}\right)^{3/2}e^{-I^e/v_{Th}^2}\right],$$
$$f^i = \frac{n_{i0}}{(2\pi)^{3/2}v_{Ti}^3}e^{-I^i/v_{Ti}^2}, \qquad v_{T\alpha}^2 = \frac{T_\alpha}{m_\alpha}, \qquad \rho = \frac{n_{h0}}{Zn_{i0}},$$
$$\mathcal{E} = \frac{e\Phi}{T_e}(1 + \Omega^2 t^2) + \frac{U^2}{2v_{Te}^2}, \tag{10}$$

where the potential Φ is defined in the implicit form

$$\frac{U^2}{2c_s^2} = \mathcal{E} - \frac{T_i}{T_i + ZT_e}\ln(1 - \rho + \rho e^{(1 - T_e/T_h)\mathcal{E}}), \tag{11}$$

and $c_s = \sqrt{(T_i + ZT_e)/(m_i + Zm_e)}$ is the ion acoustic velocity. This example may serve as a model for the expansion of a plasma bunch which was rapidly heated by a short laser pulse with significant conversion efficiency of laser energy to hot electron energy.

Example 2.—This is an expansion of a plasma bunch with different ion species having Maxwellian DFs. This example may shed light, in particular, on the role of a small number of impurity ions. Such a problem was the subject of previous research on plasma expansion into a vacuum [15]. It is of current interest because of ongoing experiments on the short-laser-pulse heating of a multispecies plasma. Numerous experiments show efficient proton acceleration because of a hydrocarbon and/or a water contamination layer on the target surfaces [16,17].

The corresponding solution of the initial-value problem reads

$$f^{\alpha} = n_{\alpha 0}(2\pi)^{-3/2} v_{T\alpha}^{-3} \exp(-I^{\alpha}/v_{T\alpha}^2). \quad (12)$$

Here Φ is given by Eq. (10), where

$$\sum_q Z_q n_{q0}\{\exp[(1 + Z_q T_e/T_{q0})\mathcal{E} - (U^2/2v_{Tq}^2)(1 + Z_q m_e/m_q)] - 1\} = 0, \qquad n_{e0} = \sum_q Z_q n_{q0}, \quad (13)$$

and $q = 1, 2, \ldots$ enumerates different species of ions. For definitiveness we choose $q = 1$ for the ion species with highest density.

Equations (10)–(13) give exhaustive information on the kinetics of plasma bunch expansion. However, for practical applications rough integral characteristics might be more useful. Two integral characteristics, such as partial ion density, $n_q(t, r)$, and ion energy spectra, $dN_q/d\epsilon$, can be calculated from the ion DFs:

$$n_q = \int_{-\infty}^{\infty} d\boldsymbol{v} \int_{-\infty}^{\infty} f^q d\boldsymbol{v}_{\perp},$$
$$\frac{dN_q}{d\epsilon} = \frac{4\pi}{m_q v} \int_0^{\infty} r^2 dr \int_{-\infty}^{\infty} (f^q(t, v) + f^q(t, -v)) d\boldsymbol{v}_{\perp}. \quad (14)$$

When integrated over the energy of radial motion, ϵ, the value $dN_q/d\epsilon$ defines the total number of ions of the given species in a plasma bunch. The ion density is defined by the universal function $\mathcal{N}_q$ as follows:

$$n_q = n_{q0}(1 + \Omega^2 t^2)^{-3/2} \mathcal{N}_q(U). \quad (15)$$

For the above examples the functions $\mathcal{N}_q(U)$ are

$$\mathcal{N}_i(U) = e^{-\mathcal{E}}(1 - \rho + \rho e^{(1 - T_e/T_h)\mathcal{E}}), \quad (16)$$

$$\mathcal{N}_q(U) = \exp\left(\frac{Z_q T_e}{T_{q0}}\mathcal{E} - \frac{U^2(1 + Z_q m_e/m_q)}{2v_{Tq}^2}\right), \quad (17)$$

respectively. The general form of $dN_q/d\epsilon$, which is given by Eqs. (7) and (14) is rather complicated but its asymptotic behavior at $\Omega t \to \infty$ is described by the simple expression

$$\frac{dN_q}{d\epsilon} \approx \frac{4\pi\sqrt{2} n_{q0}\sqrt{\epsilon_q}}{\Omega^3 m_q^{3/2}} \mathcal{N}_q\left(\frac{m_q U^2}{2} = \epsilon\right). \quad (18)$$

Here the functions $\mathcal{N}_q(U)$ are given by Eqs. (16) and (17) provided ϵ is not close to zero, $2\epsilon/T_q \gg (\Omega t)^{-2}$.

In Fig. 1 the dimensionless ion densities $\mathcal{N}_q$ are shown as the functions of the dimensionless coordinate U^2/c_s^2, where c_s is defined by the main ion component ($q = 1$). The curves in the left panel illustrate Example 1 for carbon plasma $m_i/m_e = 1836A$ ($A = 2Z = 12$) with $T_i/T_e = 0.1$, $T_h/T_e = 10$ (curve *1*), $T_h/T_e = 100$ (curve *2*), and $\rho = 0.001$. The dashed curve corresponds to plasma without hot electrons, $\rho = 0$. The curves C (carbon ions) and H (protons) in the right panel illustrate Example 2 for carbon plasma with hydrogen impurity (0.1%) where the dashed lines present the benchmarks related to pure electron-proton (*a*) and electron-carbon (*b*) plasmas.

Ion acceleration definitely depends on the electron DF [cf. line *1* (*2*) and dashed line in Fig. 1]. Even a small number of high-energy electrons provides a tenuous halo around the central plasma region. This halo expands at the sound speed determined by the temperature of hot electrons ($t \gg L_0/c_{sh}$),

$$\mathcal{N}_i \approx \rho \exp(-r^2/2c_{sh}^2 t^2), \qquad c_{sh} \approx \sqrt{Z T_h/m_i}. \quad (19)$$

The higher the temperature of hot electrons, the more energetic are the ions. The presence of ions of several types changes the dynamics of plasma bunch expansion that is demonstrated by the right panel in Fig. 1 for a plasma with light (H) and heavy (C) ions and initially Maxwellian electrons. Light ions are accelerated more efficiently and propagate ahead of the heavy ions. They form a rare halo with a density profile given by an expression similar to Eq. (19) with the ion acoustic velocity $\sqrt{Z_2 T_e/m_2}$ instead of c_{sh}.

Just as in the case of a plasma with one ion species the presence of hot electrons leads to an even higher energy of the impurity ions. This case can be studied by combining the initial conditions of Examples 1 and 2. Hot electrons significantly change the dynamics of the light impurity ions. They accelerate the impurity ions which form a halo. At $t > L_0/c_s$ this halo is in the region $r > c_{sh}t$ with the exponential density distribution as in Eq. (19), where the ion acoustic velocity, $\sqrt{Z_2 T_h/m_2}$, is defined by the mass of the impurity ion and the hot electron temperature.

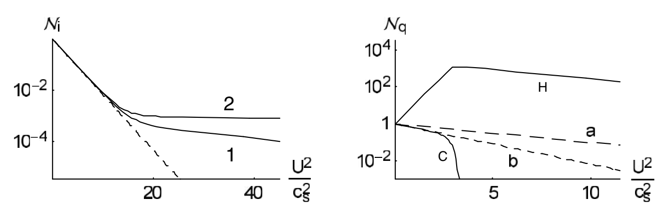

FIG. 1. The dimensionless ion densities, $\mathcal{N}_q$, for carbon plasma with (right panel) and without (left panel) hydrogen impurity.

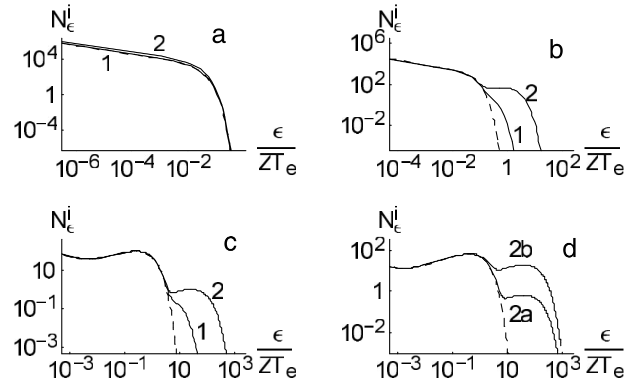

FIG. 2. The ion spectra $N_\epsilon(t, \epsilon)$ for carbon plasma with hot electrons at $\Omega t = 0$ (a), 0.1 (b), 1 (c), and 2 (d). The lines *1* and *2* correspond to $T_h/T_e = 10$ (1) and $T_h/T_e = 100$ (2) for $\rho = 0.001$ and the lines *2a* and *2b* correspond to $\rho = 0.001$ (1) and $\rho = 0.03$ (2) for $T_h/T_e = 100$.

Figure 2 shows evolution of the dimensionless ion energy spectra, $N^i_\epsilon = \sqrt{2m_i ZT_e}\Omega^3(dN_i/d\epsilon)/4\pi n_{i0}v^2_{Ti}$, for Example 1. The parameters T_i, Z, and A and the dashed line are the same as in Fig. 1. According to Eq. (18), the distribution $N^i_\epsilon(\epsilon, t)/\sqrt{\epsilon}$ definitely takes the form $\mathcal{N}_i(\epsilon)$ for $\Omega t \gg 1$. The curves *2a* and *2b* correspond to $\rho = 0.001$ and $\rho = 0.03$, respectively, to demonstrate an increase of ion acceleration efficiency with the hot electron density. Two maxima on the initially monotonic ion spectra appear with time: the first maximum at $\epsilon \approx ZT_e/2$ describes acceleration of the bulk ions, while the second one, which grows with n_h characterizes the small group of ions accelerated by hot electrons. The typical asymptotic energy of fast ions is $\epsilon = ZT_h/2$ that corresponds to the second maximum in the ion spectrum. The high end of the energy spectrum typically has a sharp decrease, so that the value $(2-3)ZT_h$ can be referred to as the characteristic ion energy cutoff.

A similar energy cutoff, $(2-3)Z_2T_h$, can be found for light impurity ions in a plasma bunch with hot electrons. Corresponding ion spectra are shown in Fig. 3 for $\Omega t = 2$ and hot electron parameters $T_h = 100T_e$ and $\rho = 0.03$. Two maxima in the ion energy spectrum are formed. The first one at the energy $\approx Z_1T_h/2$ is unique because it originates from the front of accelerated heavy ions (C). This effect, which demonstrates kinetic nature of ion acceleration in the self-consistent electric field, was discovered in particle-in-cell simulations (Fig. 7 in Ref. [18]). Acceleration of light impurity ions is of current interest in the experiments on the high-energy proton generation by short laser pulses with thin foil targets [16,17]. The measured proton energy cutoff is in qualitative agreement with the estimation $(2-3)T_h$.

The advancement in our understanding of high-energy particle production in laser-plasma interactions is dependent upon innovations in the analytical tools used. We have presented a new analytical approach which allows one to derive an entire class of three-dimensional solutions to the Cauchy problem for different initial distributions of the particles. A possible application of our results is the experiments on heating of submicron-size clusters (dusty plasma) by ultrashort laser pulses that can give rise to many potential applications.

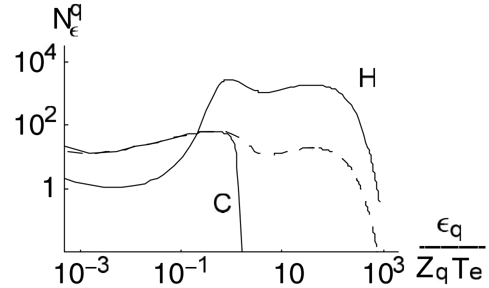

FIG. 3. The ion energy spectra for carbon plasma with (C, H) and without (dashed line) hydrogen impurity (0.1%) at $\Omega t = 2$.

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