Determination of Limit Cycles for Strongly Nonlinear Oscillators

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An innovative approach to finding limit cycles is proposed and illustrated on the van der Pol equation. The technique developed in this Letter is similar to the Ritz's method in variational theory. The present theory can be applied to not only weakly nonlinear equations, but also strongly nonlinear ones, and the obtained results are valid for the whole solution domain.

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In this study, we will consider the following nonlinear oscillation:

$$\ddot{x} + x + \varepsilon f(x, \dot{x}) = 0, \tag{1}$$

where the parameter ε needs not be small. Recently some new perturbation methods and nonperturbative methods are proposed, see, e.g., the nonperturbative method [1], the δ method [2,3], the homotopy perturbation method [4,5], the variational iteration methods [6], and the perturbation-incremental method [7,8]. A review of the recently developed analytical methods is given by myself in 2000 [9]. There also exists a wide body of literature dealing with the problem of approximate determination of limit cycles for strongly nonlinear oscillators by using various different methodologies.

In this Letter, we will propose a new technique by the variational theory (energy theory), which is quite different from the above-mentioned methods.

Generally speaking, limit cycles can be determined approximately in the form [8]

$$x = b + a(t)\cos\omega t + \sum_{n=1}^{m}(C_n \cos n\omega t + D_n \sin n\omega t), \tag{2}$$

where b, C_n, and D_n are constant.

Substituting (2) into (1) results in the following residual:

$$R(t) = \ddot{x} + x + \varepsilon f(x, \dot{x}). \tag{3}$$

In general, the residual might not be vanishingly small at all points; the error depends upon the infinite "work," dw, done by the "force" R in the infinite distance dx:

$$dw = R dx. \tag{4}$$

We hope that the total work done in a period is zero, which requires

$$\oint R dx = 0. \tag{5A}$$

Here the integral is all along the limit cycle. Under some symmetry condition, Eq. (5A) can be written in the form

$$\int_{A_0}^{A_1} R dx = 0, \tag{5B}$$

where A_0 and A_1 are the minimum and maximum amplitudes, respectively.

The above equation (5B) can be equivalently written in the form

$$\int_0^T R\dot{x} dt = 0, \tag{6}$$

where T is the period. This technique is similar to the method of weighted residuals [10].

In order to best illustrate the theory, we consider the Duffing equation as an illustrative example:

$$\ddot{x} + x + \varepsilon x^3 = 0. \tag{7}$$

Supposing that $x = a\cos\omega t + b$, where a and b are constants, we get the following residual:

$$R = a(1 - \omega^2)\cos\omega t + b + \varepsilon(a\cos\omega t + b)^3. \tag{8}$$

The constant b can be identified from the relation $b + \varepsilon b^3 = 0$, i.e., $b = 0$. So the residual can be written in the form

$$R = (1 - \omega^2)x + \varepsilon x^3. \tag{9}$$

We set

$$\int_0^a R dx = \int_0^a [(1 - \omega^2)x + \varepsilon x^3] dx = \frac{1}{2}(1 - \omega^2)a^2 + \frac{1}{4}\varepsilon a^4 = 0, \tag{10}$$

which leads to the result

$$\omega = \sqrt{1 + \frac{1}{2}\varepsilon a^2}. \tag{11}$$

We, therefore, obtain the following approximate period:

$$T = \frac{2\pi}{\sqrt{1 + 0.5\varepsilon a^2}}. \tag{12}$$

For comparison, we write the perturbation solution and the exact solution. The perturbation solution is [11]

$$T_{\text{pert}} = 2\pi(1 - \tfrac{3}{8}\varepsilon a^2), \qquad \varepsilon \gg 1, \tag{13}$$

while the exact one reads [11]

$$T_{\text{ex}} = \frac{4}{\sqrt{1+\varepsilon a^2}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k\sin^2 x}}, \qquad k = \frac{\varepsilon a^2}{2(1+\varepsilon a^2)}. \tag{14}$$

For small ε, the accuracy of our computed solution is lower than that obtained by the perturbation theory. However, our result is obtained from an energy point of view; therefore, it is valid for the whole solution domain, i.e., $0 < \varepsilon < \infty$. In case $\varepsilon \to \infty$, we have

$$\lim_{\varepsilon\to\infty} \frac{T_{\text{ex}}}{T} = \frac{2\sqrt{0.5}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - 0.5\sin^2 x}} = \frac{2\sqrt{0.5}}{\pi} \times 1.68575 = 1.19. \tag{15}$$

The 16% accuracy, when $\varepsilon \to \infty$, is remarkably good in view of the simplest trial function, $x = a\cos\omega t$. The accuracy can be dramatically improved if we choose the trial function as $x = a\cos\omega t + b\cos 3\omega t$.

Now consider an oscillator with damping term,

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0. \tag{16}$$

In such a case, the amplitude might not be a constant, but a function depending on time or frequency. So we assume

$$x = a(t)\cos\omega t + b. \tag{17}$$

By simple operation, we can identify $b = 0$. In this paper, we assume that the amplitude weakly varies with time. We write the amplitude in the form

$$a(t) = Ae^{\alpha t} \approx A, \qquad |\alpha| \ll 1. \tag{18}$$

Accordingly, we have *approximately* the following expressions:

$$\dot{x} \approx \alpha x - \omega\sqrt{A^2 - x^2}, \tag{19}$$

$$\ddot{x} \approx (\alpha^2 - \omega^2)x - 2\alpha\omega\sqrt{A^2 - x^2}. \tag{20}$$

Substituting x, $\dot{x}$, and $\ddot{x}$ into Eq. (16), we get the residual

$$R = (1 + \alpha^2 + 2\alpha\varepsilon - \omega^2)x - 2(\alpha + \varepsilon)\omega\sqrt{A^2 - x^2}. \tag{21}$$

Setting $\int_0^A R dx = 0$, we have

$$\frac{A^2}{2}(1 + \alpha^2 + 2\alpha\varepsilon - \omega^2) - \frac{1}{2}\pi A^2(\alpha + \varepsilon)\omega = 0. \tag{22}$$

After a careful examination of Eq. (19), we find that the constant, α, is a linearized factor, i.e., $\dot{x} \sim \alpha x$. This analysis leads to the identification of the constant α from the following relation:

$$\dot{x} = -\frac{1}{2\varepsilon}(x + \ddot{x}) \sim \alpha x. \tag{23}$$

Omitting the nonlinear terms, we have

$$\alpha = -\frac{1}{2\varepsilon}(1 + \alpha^2 - \omega^2). \tag{24}$$

In view of Eqs. (22) and (24), we finally obtain

$$\alpha = -\varepsilon \quad \text{and} \quad \omega = \sqrt{1 - \varepsilon^2}. \tag{25}$$

So we have the following solution:

$$x = Ae^{-\varepsilon t}\cos(\sqrt{1 - \varepsilon^2})t, \tag{26}$$

which is an exact solution.

The above illustration is important to nonlinear oscillators with damping terms. Now we consider the van der Pol equation:

$$\ddot{x} + x - \varepsilon(1 - x^2)\dot{x} = 0. \tag{27}$$

Substituting Eqs. (17), (19), and (20) into Eq. (27), and noting that the constant, b, is easily identified as $b = 0$, we have the following residual:

$$R = (1 + \alpha^2 - \omega^2)x - 2\alpha\omega\sqrt{A^2 - x^2} - \varepsilon(1 - x^2)(\alpha x - \omega\sqrt{A^2 - x^2}). \tag{28}$$

By similar operation as illustrated above, we set $\int_0^A R dx = 0$, i.e.,

$$\frac{A^2}{2}(1 + \alpha^2 - \omega^2 - \varepsilon\alpha) - \frac{1}{4}\pi A^2\omega(2\alpha - \varepsilon) + \frac{1}{4}\ \mathrm{A}^4\varepsilon\alpha - \frac{1}{4}A^4\varepsilon\omega = 0. \tag{29}$$

Note that when

$$\dot{x} = \frac{\ddot{x} + x}{\varepsilon(1 - x^2)} \sim \alpha x, \tag{30}$$

we have

$$1 - \alpha^2 - \omega^2 - \varepsilon\alpha = 0. \tag{31}$$

In view of Eqs. (29) and (31), we obtain the following relation:

$$\alpha = -\frac{(A^2 - \pi)\omega\varepsilon}{2\pi\omega - \varepsilon A^2}. \tag{32}$$

Substituting α into (31) yields the result

$$1 - \frac{\varepsilon^2(A^2 - \pi)^2\omega^2}{(2\pi\omega - \varepsilon A^2)^2} - \omega^2 + \frac{\varepsilon^2(A^2 - \pi)\omega}{2\pi\omega - \varepsilon A^2} = 0. \tag{33}$$

In case $\varepsilon \ll 1$, the above equation reduces to

$$\omega = 1 - \frac{\varepsilon^2(A^2 - \pi)^2}{8\pi^2} + \frac{\varepsilon^2(A^2 - \pi)}{4\pi} + O(\varepsilon^3). \tag{34}$$

When $\varepsilon \to \infty$, from Eq. (33), we obtain

$$\omega = \frac{A^2}{(A^2 - \pi)\varepsilon} + O\left(\frac{1}{\varepsilon^2}\right). \quad (35)$$

Perturbation method shows that $A = 2$, so we have the following solution:

$$x = Ae^{-\alpha t}\cos\omega t = 2\exp\left(-\frac{(4-\pi)\omega\varepsilon t}{2\pi\omega - 2\varepsilon}\right)\cos\omega t \quad (36)$$

The frequency is determined from Eq. (33). In case $\varepsilon \to \infty$, we obtain the period

$$T = \frac{2\pi}{\omega} = \frac{2\pi(A^2 - \pi)\varepsilon}{A^2} = 1.348\varepsilon, \quad (37)$$

while the exact one is $T_{\rm ex} = 1.614\varepsilon$ when $\varepsilon \to \infty$. The 16.5% accuracy is good as the obtained solution is valid for the whole solution domain $0 \le \varepsilon < \infty$.

Lopez-Ruiz and Lopez [12] calculated exactly the limit cycles of some Lienard systems in the weakly ($\varepsilon \to 0$) and in the strongly ($\varepsilon \to \infty$) nonlinear regimes. Our results agree with Lopez-Ruiz and Lopez's exact solution in the weakly and strongly nonlinear regimes.

By the homotopy perturbation method [4,5], and the modified Lindsted-Poincare method [13–15], the present author obtained the following approximate period for the van der Pol equation:

$$T = 2\pi\sqrt{1 + \frac{1}{8}\varepsilon^2}. \quad (38)$$

The relative error is 37.5% when $\varepsilon \to \infty$.

In summary, we conclude from the results obtained that the method developed here is extremely simple in its principle, quite easy to use, and gives a very good accuracy in the whole solution domain, even with the simplest trial functions. The accuracy depends upon the trial functions similar to that in the Ritz's method. Theoretically, any accuracy can be achieved by a suitable choice of the trial functions. Coupled with other known results, the method provides a powerful mathematical tool to the determination of limit cycles.

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