Global Charges of Stationary Non-Abelian Black Holes

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We consider stationary axially symmetric black holes in SU(2) Einstein-Yang-Mills-dilaton theory. We present a mass formula for these stationary non-Abelian black holes, which also holds for Abelian black holes. The presence of the dilaton field allows for rotating black holes, which possess nontrivial electric and magnetic gauge fields, but do not carry a non-Abelian charge. We further present a new uniqueness conjecture.

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Introduction.—Black holes in Einstein-Maxwell (EM) theory are uniquely characterized by their global charges: their mass M, their angular momentum J, their electric charge Q, and their magnetic charge P [1,2]. EM black holes further satisfy remarkable relations between their horizon properties and their global charges [2], such as the Smarr formula [3],

$$M = 2TS + 2\Omega J + \Psi_{\rm el,H} Q + \Psi_{\rm mag,H} P, \tag{1}$$

where T represents the temperature of the black holes and S their entropy, Ω denotes their horizon angular velocity, and $\Psi_{\rm el,H}$ and $\Psi_{\rm mag,H}$ represent their horizon electrostatic and magnetic potential, respectively.

When non-Abelian gauge fields are coupled to gravity, black hole solutions are no longer uniquely characterized by their global charges [4,5], and neither does the mass formula, Eq. (1), hold [5,6]. SU(2) Einstein-Yang-Mills (EYM) theory, for instance, possesses sequences of black hole solutions, which carry non-Abelian magnetic fields outside their regular horizon, but no non-Abelian magnetic charge [4,7–9]. These non-Abelian black hole solutions are characterized additionally by the node number k and the winding number n of their gauge fields [4,7].

In many unified theories, including Kaluza-Klein theory and string theory, a scalar dilaton field arises naturally. Black holes in Einstein-Maxwell-dilaton (EMD) theory [10–12] and in SU(2) Einstein-Yang-Mills-dilaton (EYMD) theory [7,13], possess a further global charge, the dilaton charge D, describing the asymptotic falloff of the dilaton field. For EMD black holes with vanishing electromagnetic charges the dilaton charge vanishes as well. In contrast, the dilaton charge is finite for all genuinely non-Abelian black holes of EYMD theory.

Here we utilize the dilaton charge, to derive a mass formula for stationary axially symmetric SU(2) EYMD black holes, similar to the Smarr formula,

$$M = 2TS + 2\Omega J + 2\Psi_{\rm el,H} Q + \frac{D}{\gamma}, \tag{2}$$

where γ is the dilaton coupling constant. In this new mass formula the dilaton charge D enters, instead of the magnetic charge P, present in the Smarr formula. This is important, since the genuinely non-Abelian black hole solutions carry magnetic fields, but no magnetic charge. Thus the dilaton charge term takes into account the contribution to the total mass from the magnetic fields outside the horizon. This mass formula holds for all nonperturbatively known black hole solutions of SU(2) EYMD theory [7,9,13], including the rotating generalizations of the static nonspherically symmetric non-Abelian black hole solutions, presented here.

Non-Abelian black holes.—We consider black holes of the SU(2) EYMD action with matter Lagrangian

$$4\pi L_M = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} e^{2\gamma\phi} {\rm Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{3}$$

For the metric we choose the stationary axially symmetric Lewis-Papapetrou metric in isotropic coordinates,

$$ds^2 = -f dt^2 + \frac{m}{f}(dx^2 + x^2 d\theta^2) + \frac{l}{f} x^2 \sin^2\theta \left(d\varphi - \frac{\omega}{x} dt \right)^2 \tag{4}$$

with Killing vectors $\xi = \partial_t$ and $\eta = \partial_\varphi$. For the gauge field $A = A_\mu dx^\mu$ symmetry requires $(\mathcal{L}_\xi A)_\mu = D_\mu W_\xi$, $(\mathcal{L}_\eta A)_\mu = D_\mu W_\eta$ with compensating su(2)-valued functions W_ξ and W_η. We employ the generalized ansatz,

$$A = \psi dt + A_\varphi \left(d\varphi - \frac{\omega}{x} dt \right) + \left[\frac{H_1}{x} dx + (1 - H_2) d\theta \right] \frac{\tau_\varphi^n}{2}, \tag{5}$$

$$A_\varphi = -n\sin\theta\left[H_3\frac{\tau_r^n}{2} + (1-H_4)\frac{\tau_\theta^n}{2}\right],$$
$$\psi = B_1\frac{\tau_r^n}{2} + B_2\frac{\tau_\theta^n}{2},$$

with $\tau_r^n = \vec{\tau}\cdot(\sin\theta\cos n\varphi, \sin\theta\sin n\varphi, \cos\theta)$, etc., and n is the winding number of the solutions. For this ansatz $W_\xi = 0$ and $W_\eta = n\tau_z/2$. All functions depend only on x and θ.

The event horizon of stationary black holes resides at a surface of constant radial coordinate $x = x_{\rm H}$, and is characterized by the condition $f(x_{\rm H}) = 0$. The Killing vector field $\chi = \xi + \Omega\eta$ is orthogonal to and null on the horizon, where Ω is the horizon angular velocity [14]. At the horizon we impose the boundary conditions [9] $f = m = l = 0$, $\omega = \omega_{\rm H} = \Omega x_{\rm H}$, $\partial_x\phi = 0$, $H_1 = 0$, $\partial_x H_2 = \partial_x H_3 = \partial_x H_4 = 0$, $B_1 - n\Omega\cos\theta = 0$, $B_2 + n\Omega\sin\theta = 0$.

Axial symmetry and regularity impose the boundary conditions on the symmetry axis ($\theta = 0$), $\partial_\theta f = \partial_\theta l = \partial_\theta m = \partial_\theta\omega = 0$, $\partial_\theta\phi = 0$, $H_1 = H_3 = B_2 = 0$, $\partial_\theta H_2 = \partial_\theta H_4 = \partial_\theta B_1 = 0$, and agree with the boundary conditions on the $\theta = \pi/2$ axis, except for $B_1 = 0$, $\partial_\theta B_2 = 0$.

The boundary conditions at infinity, $f = m = l = 1$, $\omega = 0$, $\phi = 0$, $H_1 = H_3 = 0$, $H_2 = H_4 = (-1)^k$, $B_1 = B_2 = 0$, where k denotes the node number, ensure that the black holes are asymptotically flat and magnetically neutral.

The global charges of the EYMD black holes are obtained from the asymptotic expansion. The metric functions yield the mass M, and the angular momentum $J = aM$,

$$f \to 1 - \frac{2M}{x}, \qquad \omega \to \frac{2J}{x^2}, \tag{6}$$

the matter functions yield the non-Abelian electric charge Q, and the dilaton charge D,

$$B_1 \to \frac{Q\cos\theta}{x}, \qquad B_2 \to -(-1)^k\frac{Q\sin\theta}{x}, \qquad \phi \to -\frac{D}{x}. \tag{7}$$

The definition of Q corresponds to rotating to a gauge, where $\psi \to \frac{Q}{x}\frac{\tau_z}{2}$ [9] (see also [15]). Note, that the modulus of the non-Abelian electric charge, $|Q|$, agrees with the gauge invariant definition of the non-Abelian electric charge given in Ref. [5].

Mass formula.—We derive the mass formula, Eq. (2), both for genuinely non-Abelian black holes, satisfying the above set of boundary conditions, and for embedded Abelian black holes, possessing both electric and magnetic charges [9,12,16]. We embed the Abelian black holes via

$$A_\mu dx^\mu = (\mathcal{A}_0 dt + \mathcal{A}_\varphi d\varphi)\frac{\tau_z}{2}, \tag{8}$$

where $\mathcal{A}$ denotes the Abelian gauge field.

We start from the general expression for the mass of stationary axially symmetric black holes [14],

$$M = 2TS + 2\Omega J_{\rm H} - \frac{1}{4\pi}\int_\Sigma R_0^0\sqrt{-g}\,dx\,d\theta\,d\varphi, \tag{9}$$

and express R_0^0 with the help of the Einstein equations and the dilaton equation of motion,

$$R_0^0 = -\frac{1}{\gamma}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,\partial^\mu\phi) + 4e^{2\gamma\phi}{\rm Tr}(F_{0\alpha}F^{0\alpha}). \tag{10}$$

Evaluating the integral involving the dilaton d'Alembertian we obtain the dilaton term, D/γ, in the mass formula [7].

We replace the horizon angular momentum $J_{\rm H}$ by the global angular momentum J [14],

$$J = J_{\rm H} + \frac{1}{2\pi}\int_\Sigma e^{2\gamma\phi}{\rm Tr}(F_{\varphi\alpha}F^{0\alpha})\sqrt{-g}\,dx\,d\theta\,d\varphi, \tag{11}$$

and obtain

$$M - 2TS - 2\Omega J - \frac{D}{\gamma} = I \equiv -\frac{1}{\pi}\int_\Sigma e^{2\gamma\phi}{\rm Tr}[(F_{0\alpha} + \Omega F_{\varphi\alpha})F^{0\alpha}]\sqrt{-g}\,dx\,d\theta\,d\varphi. \tag{12}$$

To evaluate the remaining integral I, we make the replacements $F_{\alpha 0} = D_\alpha A_0$ and $F_{\alpha\varphi} = D_\alpha(A_\varphi - W_\eta)$ [17], where $D_\alpha = \partial_\alpha + i[A_\alpha, \cdot]$. For embedded Abelian solutions, $W_\eta = 0$. Employing the gauge field equations of motion, we obtain

$$I = \frac{1}{\pi}\int_\Sigma {\rm Tr}[D_\alpha(\Psi - \Omega W_\eta)e^{2\gamma\phi}F^{0\alpha}\sqrt{-g})]\,dx\,d\theta\,d\varphi, \tag{13}$$

with electrostatic potential $\Psi = \chi^\mu A_\mu = A_0 + \Omega A_\varphi$. We replace the gauge covariant derivative by the partial derivative, since the trace of a commutator vanishes. Only the $\alpha = r$ term contributes to the integral I [18],

$$I = \frac{1}{\pi}\int {\rm Tr}[(\Psi - \Omega W_\eta)e^{2\gamma\phi}F^{0r}\sqrt{-g}]|_{x_{\rm H}}^{\infty}\,d\theta\,d\varphi. \tag{14}$$

Thus the mass formula holds, provided $I = 2\Psi_{\rm el,H}Q$.

We first prove this equality for embedded Abelian black holes. We introduce the Abelian electrostatic potential, $\Psi_{\rm el} = \chi^\mu\mathcal{A}_\mu = \mathcal{A}_0 + \Omega\mathcal{A}_\varphi$, and the conserved charge $\tilde{Q}$ [19],

$$\tilde{Q} = -\frac{1}{4\pi}\int_S e^{2\gamma\phi}(^*\mathcal{F}_{\theta\varphi})d\theta d\varphi. \quad (15)$$

$\tilde{Q}$ does not depend on the choice of two-sphere S, i.e., $\tilde{Q}(x_H) = \tilde{Q}(\infty) = \tilde{Q}$, and $\tilde{Q} = Q$ for $\phi(\infty) = 0$. Employing the asymptotic expansion, $\mathcal{F}^{0r}\sqrt{-g} = -Q\sin\theta + o(1)$, and noting that the electrostatic potential is constant at the horizon, the integral I becomes

$$I = 2Q\left(\Psi_{\rm el,H} - \frac{1}{4\pi}\int \Psi_{\rm el,\infty}\sin\theta d\theta d\varphi\right). \quad (16)$$

$\Psi_{\rm el}$ is determined only up to a constant. In a gauge where the contribution to I from infinity vanishes, we obtain $I = 2\Psi_{\rm el,H}Q$. On the other hand, in a gauge where $\Psi'_{\rm el,H} = 0$ the integral receives its only contribution from infinity, and $\frac{1}{4\pi}\int \Psi'_{\rm el,\infty}\sin\theta d\theta d\varphi = -\Psi_{\rm el,H}$, yielding again $I = 2\Psi_{\rm el,H}Q$. When the Smarr formula holds [12,20], the mass formula Eq. (2) implies $D/\gamma = \Psi_{\rm mag,H}P - \Psi_{\rm el,H}Q$.

For genuinely non-Abelian black holes the equations of motion still require the electrostatic potential to be constant at the horizon. However, in contrast to the Abelian case, where the horizon electrostatic potential is only determined up to a constant, genuinely non-Abelian black holes are only obtained if $\Psi_H = \Omega W_\eta$ [9]. This condition is invariant under time independent gauge transformations, and it is implemented via the boundary conditions. At the same time this condition leads to the vanishing of the integrand of I at the horizon. Hence, for non-Abelian solutions, the integral I receives a contribution always only from infinity.

Subject to the above ansatz and boundary conditions, we see from the asymptotic expansion that at infinity $F^{0r}\sqrt{-g} = -Q\sin\theta[\cos\theta\tau_r^n - (-1)^k\sin\theta\tau_\theta^n]/2 + o(1)$ remains finite, $A_0 = o(1)$ does not contribute to the integral, and $A_\varphi = -n[1-(-1)^k]\sin\theta\tau_\theta^n/2 + o(1)$ contributes only for odd node number k. Taking the trace leaves the same angular integral, independent of k, yielding $I = 2\Omega nQ = 2\Psi_{\rm el,H}Q$ with $\Psi_{\rm el,H} = \Omega n$ [21]. This completes the proof for genuinely non-Abelian black holes.

The numerically constructed stationary axially symmetric EYMD black holes satisfy the mass formula, Eq. (2), with an accuracy of 10^{-3}. So do the numerically constructed EMD black holes. The EYM black holes are included in the limit $\gamma \to 0$, since D/γ remains finite.

Stationary $Q = 0$ solutions.—While similar in many respects to EYM black holes [9], EYMD black holes also possess new features. In Fig. 1 we exhibit the global charges, the mass M, the specific angular momentum a, the non-Abelian electric charge Q, and the relative dilaton charge D/γ as functions of the dilaton coupling constant γ for $k = n = 1$ black hole solutions. Interestingly, the non-Abelian electric charge Q can change sign in the presence of the dilaton field.

Thus we observe the surprising feature that the non-Abelian charge Q of rotating EYMD black holes can vanish. Cuts through the parameter space of solutions with vanishing Q are exhibited in Fig. 2. Rotating black hole solutions with $Q = 0$ exist only above a minimal value of γ; $\gamma_{\rm min} \approx 1.15$ for $k = n = 1$. These $Q = 0$ EYMD black holes represent the first black hole solutions, which carry nontrivial non-Abelian electric and magnetic fields and no non-Abelian charge. As a consequence, these special solutions do not exhibit the generic asymptotic noninteger power falloff of the stationary non-Abelian gauge field solutions [9,20].

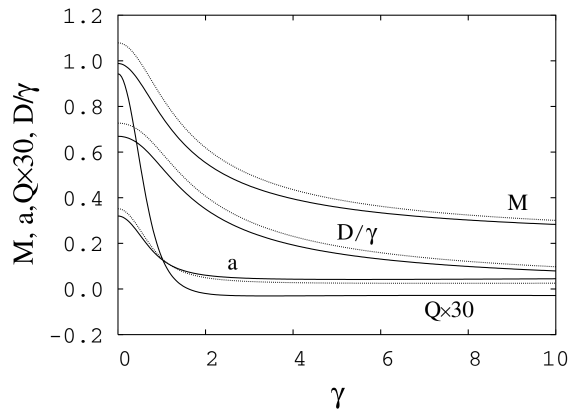

FIG. 1. The global charges are shown as functions of the dilaton coupling constant γ ($x_H = 0.1$, $\omega_H = 0.02$, $k = n = 1$). Also shown are the global charges of embedded Abelian solutions with $Q = 0$ and $P = 1$ (dotted).

Uniqueness conjecture.—Reconsidering the uniqueness conjecture for EYMD black holes, we replace the magnetic charge P by the dilaton charge D. However, black holes in SU(2) EYMD theory are not uniquely characterized by their mass M, their angular momentum J, their non-Abelian electric charge Q, and their dilaton charge D. This is seen in Fig. 3, where the relative dilaton charge D/γ is shown as a function of the mass M, for the static non-Abelian black hole solutions with $n = 1 - 3$, $k = 1 - 3$ ($\gamma = 1$). Whereas solutions with the same winding number n do not intersect, those with different

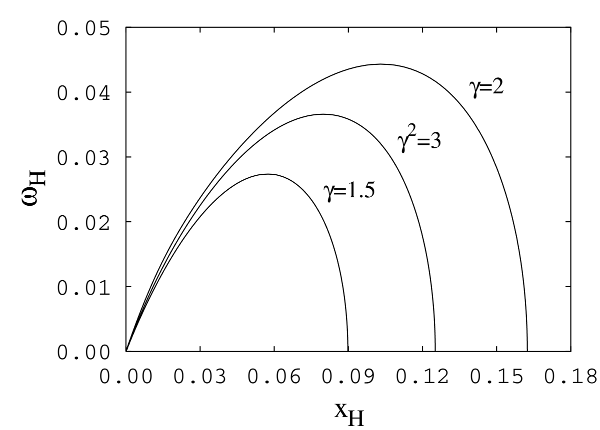

FIG. 2. Cuts through the parameter space of $Q = 0$ black hole solutions ($\gamma = 1.5$, $\sqrt{3}$, 2, $k = n = 1$).

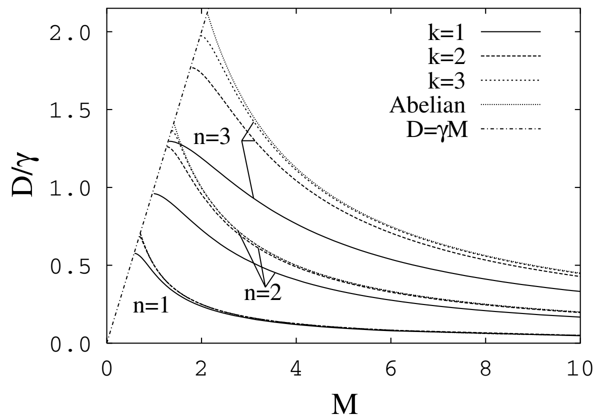

FIG. 3. The relative dilaton charge D/γ is shown as a function of the mass for static non-Abelian solutions ($\gamma = 1$) and embedded Abelian solutions with $Q = 0$ and $P = n$.

winding numbers can intersect. We therefore introduce an additional charge, a topological charge N suggested by Ashtekar [22],

$$N = \frac{1}{4\pi} \int_{\mathrm{H}} \frac{1}{2} \varepsilon_{ijk} \sigma^i d\sigma^j \wedge d\sigma^k, \qquad (17)$$

where σ is a map from the horizon two-sphere H to the two-sphere of directions in the Lie-algebra of SU(2), defined from the pullback of the Yang-Mills field strength to H, $F_{\mathrm{H}} = F_{\theta\phi}|_{\mathrm{H}} d\theta \wedge d\phi$. For non-Abelian solutions, $N = n$, for embedded Abelian solutions, $N = 0$.

We thus put forward a new uniqueness conjecture, stating that *black holes in SU(2) EYMD theory are uniquely determined by their mass M, their angular momentum J, their non-Abelian electric charge Q, their dilaton charge D, and their topological charge N*. To illustrate the validity of the conjecture also for stationary black holes, we show in Fig. 4 the relative dilaton charge

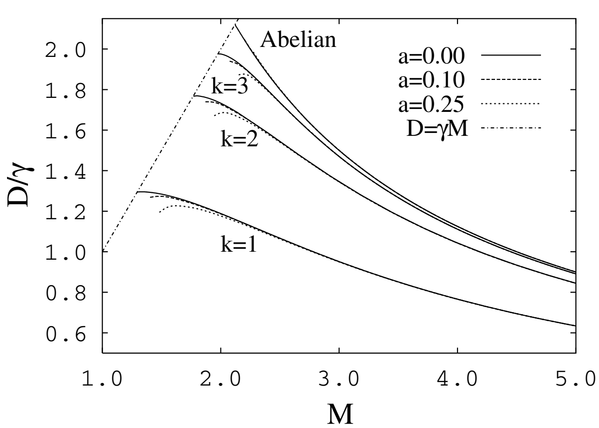

FIG. 4. The relative dilaton charge D/γ is shown as a function of the mass for non-Abelian black holes with $n = 3$, ($\gamma = 1$), and embedded Abelian solutions with $Q = 0$ and $P = 3$, for specific angular momentum $a = 0.25$, 0.1, and 0.

D/γ for several values of the specific angular momentum a for black holes with $n = 3$, $k = 1 - 3$ ($\gamma = 1$).

Since the relative dilaton charge D/γ remains finite in the limit $\gamma \to 0$, this conjecture may formally be extended to the EYM case, by replacing the dilaton charge D by the relative dilaton charge D/γ.

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