Bose-Fermi Mixtures in One Dimension

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We analyze the phase stability and the response of a mixture of bosons and spin-polarized fermions in one dimension (1D). Unlike in 3D, phase separation happens for low fermion densities. The dynamics of the mixture at low energy is independent of the spin-statistics of the components, and the modes are essentially undamped.

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Binary mixtures of dilute quantum gases are a subject of steadily growing interest initiated by the realization of Bose-Einstein condensation (BEC) of alkali atoms [1] and motivated by the quest for and subsequent experimental realization of degenerate Fermi gas [2]. Strong s-wave interactions that facilitate evaporative cooling of bosons are absent among spin-polarized fermions due to the exclusion principle; so fermions are cooled to degeneracy through the mediation of fermions in another spin state [2] or via a buffer gas of bosons [3,4]. Degeneracy in dilute gases can be understood better than that in liquid helium due to weaker interactions and, thus, offers prospects of detailed quantitative study of several interesting phenomena in the physics of many-body quantum systems such as the Bardeen-Cooper-Schrieffer transition.

On another front, a new generation of BEC experiments on surface microtraps [5,6] and experiments on creating atomic waveguides [7] have generated interest in quantum gases in lower dimensions. Effective 1D and 2D BECs have been created, in which excitations in the confined directions are energetically not allowed [8]. BEC on optical lattices [9] are being actively studied; the atoms at each lattice site can be in effective 1D.

It is therefore a natural step to bring these two exciting developments together and consider binary mixtures of quantum gases in effective 1D, with the possibility of forming one-dimensional degenerate Fermi gases and fermionic waveguides. Fermions in 1D have been the subject of some seminal models in many-body quantum physics [10], because they are theoretically more tractable than in 3D. Now there is the possibility of testing these models experimentally. Considerable recent theoretical work has been done on Bose-Fermi mixtures (BFM) in 3D [11–14] but little has been said about 1D systems. The goal of this Letter is to study theoretically some of the properties of binary mixtures of bosons and spin-polarized fermions in an effective 1D configuration. In particular, we consider their excitations and phase stability.

Model.—We consider a longitudinally homogeneous 1D mixture of N_b hard-core bosons of mass m_b and N_f spin-polarized fermions of mass m_f at $T = 0$ K. A natural choice of trap-geometry to consider such a mixture is in a toroidal trap [15] with no external potential along the circumference (of length L), but with tight cylindrically symmetric harmonic confinement of frequency ω_0 in the transverse direction. For the atoms to have effective 1D behavior at $T = 0$ K, the ground state energy of the transverse trapping potential has to exceed the ground state energy of the bosons and the fermions in 3D, i.e., $\hbar\omega_0 \gg \mu_{b(3D)}$ and $\hbar\omega_0 \gg \epsilon_{f(3D)}$, where $\mu_{b(3D)} = 4\pi\hbar^2 a_b n_{b(3D)}/m_b$ is the bosonic chemical potential and $\epsilon_{f(3D)} = \hbar^2(6\pi^2 n_{f(3D)})^{2/3}/2m_f$ is the Fermi energy of noninteracting fermions, both in 3D. A measure of the transverse spread of the atoms in the ring is given by the single-particle ground state widths for the transverse trap, $r_{b(f)} = \sqrt{\hbar/m_{b(f)}\omega_0}$. A torus of high aspect ratio has $L \gg r_{b(f)}$. We define twice the reduced mass $m_{bf} = 2m_f m_b/(m_f + m_b)$.

To describe the 1D BFM, we use an effective Hamiltonian for its longitudinal behavior in the toroidal trap

$$\hat{H} = \int dx \hat{\psi}_b^\dagger\Big[-\frac{\hbar^2}{2m_b}\partial_x^2 - \mu_b + \frac{g_b}{2}\hat{\psi}_b^\dagger\hat{\psi}_b\Big]\hat{\psi}_b + \int dx\hat{\psi}_f^\dagger\Big[-\frac{\hbar^2}{2m_f}\partial_x^2 - \mu_f\Big]\psi_f + g_{bf}\int dx\hat{\psi}_b^\dagger\hat{\psi}_b\psi_f^\dagger\psi_f. \quad (1)$$

Here $\hat{\psi}_{b(f)}(x, t)$ are field operators for the longitudinal degree of freedom and x is the circumferential spatial coordinate. In assuming noninteracting fermions, we neglected weak p-wave or higher binary interactions. We also assume factorization of the transverse degrees of freedom, justified in effective 1D [16] since the transverse spatial profile is that of the single-particle ground states for the trap potential regardless of the longitudinal behavior or statistics. Integrating over the transverse coordinates [16] gives the effective 1D coupling strengths for the boson-boson and boson-fermion interactions: $g_b = 2\hbar\omega_0 a_b$ and $g_{bf} = 2\hbar\omega_0 a_{bf}$, with respective scattering lengths a_b and a_{bf}. The linear density operators are $\hat{\rho}_{b(f)}(x, t) = \hat{\psi}_{b(f)}^\dagger(x, t)\hat{\psi}_{b(f)}(x, t)$, with spatially constant equilibrium expectations

$n_{b(f)} = N_{b(f)}/L$ and fluctuation operators $\hat{\delta}\rho_{b(f)}(x, t) = \hat{\rho}_{b(f)}(x, t) - n_{b(f)}$.

Phase stability in the static limit.—We first consider the mixture in static equilibrium when the expectations of the fluctuation operators are zero; the kinetic energy of the bosons vanishes while the kinetic energy for the fermions contributes the Fermi energy per particle $\epsilon_f = g_f n_f^2/3$, with $g_f = \hbar^2\pi^2/2m_f$ and the Fermi wave vector $k_f = \pi n_f$. In this case the *total* particle number is fixed, so we take the ground state expectation of the *canonical* Hamiltonian for the system and thus obtain a simple expression for the total energy of a *uniform* mixture of bosons and fermions at equilibrium,

$$E_u = L[(1/2)g_b n_b^2 + (1/3)g_f n_f^3 + g_{bf} n_b n_f]. \quad (2)$$

Its derivatives with respect to the densities yield

$$\mu_b = g_b n_b + g_{bf} n_f, \qquad \mu_f = g_{bf} n_b + g_f n_f^2, \quad (3)$$

the Thomas-Fermi equations for the chemical potentials. The derivative with respect to L gives the pressure

$$p = -\partial E/\partial L = (1/2)g_b n_b^2 + (2/3)g_f n_f^3 + g_{bf} n_b n_f. \quad (4)$$

The second derivative condition for a stable minimum with respect to small changes in the densities requires

$$n_f \geq g_{bf}^2/(2g_f g_b) = 2a_{bf}^2/\pi^2 a_b r_f^2, \quad (5)$$

which puts a *lower limit* on the fermion density, *opposite* to the constraint in 3D where the stability condition puts an *upper limit* on the fermion density. The reason for the difference is that the power law of the density dependency of the Fermi energy changes with dimensionality. The energy due to the Fermi pressure grows faster as a function of linear density in 1D than it does with increase in bulk density in 3D; however, the boson-fermion interaction energy behaves similarly in 3D and in 1D with respect to bulk density and linear density, respectively; thus, at higher fermion densities, the total energy in 1D is more likely to be lowered if the fermions are spread out over a larger volume mixed in with the bosons.

The stability criterion in (5) applies for small fluctuations; we now analyze the general phase stability for a BFM in 1D, as was done for 3D mixtures by Viverit *et al.* [11]. A binary mixture can have at most two distinct phases; we use subscripts $i = 1, 2$ to label the physical quantities specific to each. The volume fractions of the phases are $\ell = L_1/L$ and $1 - \ell = L_2/L$ and the ratio of the densities in the two phases $\eta_{b(f)} = n_{b(f),1}/n_{b(f),2}$. The total energy for the phase-separated mixture is

$$E_s = \sum_{i=1}^{2} L_i E_i = \sum_{i=1}^{2} L_i\left[\frac{g_b}{2}n_{b,i}^2 + \frac{g_f}{3}n_{f,i}^3 + g_{bf} n_{b,i} n_{f,i}\right]. \quad (6)$$

Equilibrium between the phases requires

$$p_1 = p_2; \quad (7a)$$

$$\mu_{b(f),1} = \mu_{b(f),2}, \quad (7b)$$

$$\mu_{b(f),i} > \mu_{b(f),j} \qquad \text{if } n_{b(f),i} = 0, \quad (7c)$$

where the pressure and chemical potentials in each phase are given by Eqs. (3) and (4) with the total densities *replaced* by partial densities. We use the identity $\ell n_{b(f),1} + (1-\ell)n_{b(f),2} = n_{b(f)}$ in Eq. (2) to evaluate the energy E_u of the uniform phase to compare with the energy E_s of the phase-separated mixture in Eq. (6). We do the comparison by calculating $\Delta = [E_u - E_s]/L$, using density measures $C_f = g_{bf}^2/g_f g_b$ for fermions and $C_b = g_{bf}^3/g_f g_b^2$ for bosons. There are four possible ways of phase separation; the feasibility of each is determined by the specific nature of the conditions (7) and the principle of minimum energy. We now discuss each case, leaving out the algebra.

(i) Two pure phases: The fermions are all in one phase and the bosons in the other, so we set $n_{f,1} = n_{b,2} = 0$. The equilibrium conditions (7a) and (7c) constrain the partial densities: $n_{f,2} \leq 3C_f/4$ and $n_{b,1} \leq 3C_b/4$. When those conditions are used in Eqs. (2) and (6), they give $\Delta \geq (1-\ell)\ell^2 g_F n_{F,2}^3/3 \geq 0$, which means that the separated phase has lower energy for all values of the volume fraction $\ell \in [0, 1]$ and hence is energetically preferred in the density regimes where phase equilibrium is possible, $n_f \leq (3/4)(1-\ell)C_f$ and $n_b \leq (3/4)\ell C_b$.

(ii) A mixed phase and purely bosonic phase: The fermions are all in one phase, $n_{f,1} = 0$, but there are bosons in both phases. Equations (7a) and (7b) fix the fermion partial density $n_{f,2} = 3C_f/4$, which satisfies the condition $n_{f,2} \leq C_f$ imposed by the inequality of the fermion chemical potentials $\mu_{f,1} \geq \mu_{f,2}$. Then it follows from $\mu_{b,1} = \mu_{b,2}$ that $n_{b,1} - n_{b,2} = 3C_b/4$.

On applying Eqs. (7a) and (7b) to Eqs. (2) and (6), we find $\Delta = (1-\ell)\ell^2 g_F n_{F,2}^3/3 \geq 0$, so that the separated phase is energetically preferred in this case as well, in the density regimes $n_f = (3/4)(1-\ell)C_f$ and $n_b = n_{b,2} + (3/4)\ell C_b$, which do not overlap with those of the previous case.

(iii) A mixed phase and purely fermionic one: All the bosons are in one phase, $n_{b,2} = 0$, while the fermions can be in both phases. The phase equilibrium conditions require that the fermion density ratio $\eta_f \in [0, 1]$ and that it satisfies the equation $3(n_{f,2}/C_f)(1+\eta_f)^2 = 2(2+\eta_f)$. This limits the fermion density in the second phase to be $C_f/2 \leq n_{f,2} \leq 4C_f/3$. But Eq. (7a) and $\mu_{f,1} = \mu_{f,2}$ imply $\Delta = \ell(1-\ell)^2 g_F[n_{F,1} - n_{F,2}]^3/3 \leq 0$, where the inequality holds because $\eta_f \leq 1$. The uniform mixture will thus be energetically preferred for all values of ℓ; hence, this type of phase separation will not occur.

(iv) Two mixed phases: Both phases have fermions as well as bosons. The three equations arising from the equilibrium conditions (7a) and (7b) lead to an equation for the fermion density ratio, $(1-\eta_f)^3 = 0$ with the only solution $\eta_f = 1$. Because $\mu_{b,1} = \mu_{b,2}$, this implies that the boson density ratio $\eta_b = 1$. Thus, the only allowed solution is when the entire system is uniform, and so there is no phase separation of this type either.

The allowed phases for different linear densities are plotted in Fig. 1; phase separation occurs for low fermion densities, in qualitative agreement with Eq. (5). Taking the bulk density to be $n_{f(3D)} \simeq n_f/\pi r_f^2$, the criteria for single phase $n_f > \frac{3}{4}C_f$ and effective 1D $\epsilon_{f(3D)} \ll \hbar\omega_0$ give the limits of fermion density for which bosons and fermions in 1D can coexist in a single phase: $a_{bf}^2/a_b r_f^2 < n_f < 1/r_f$. Transverse trap widths $r_f \sim 1$ μm achievable currently would allow single phase mixtures for densities up to $n_{f(3D)} \sim 10^{18}$ m^{-3}, which was that of the coldest ^{6}Li samples in a recent experiment [17] that created a degenerate system of bosons (^{23}Na) and fermions (^{6}Li) in 3D. Scattering lengths for alkali atoms are ~1–10 nm, which allows a range of few orders of magnitude of fermion density where a *stable uniform* 1D BFM would form; this range can be widened by strengthening the transverse trap or reducing the boson-fermion scattering length. Unlike in 3D, phase separation effects in 1D can be observed by reducing the density, which is usually easier to do than increasing it.

Dynamic response.—We now consider the dynamical properties of the mixture. For weak interaction strengths and low energy modes, we can use linear response theory. We consider small density fluctuations of the bosons and the fermions about equilibrium. The boson fluctuation can be considered a density dependent perturbation for the fermions and vice versa, so we have two coupled linear equations for the Fourier transforms of the expectation of the density fluctuations $\delta\rho_b(x, t)$ and $\delta\rho_f(x, t)$,

$$\begin{aligned} \delta\rho_b(q, \omega) &= \chi_b(q, \omega) g_{bf} \delta\rho_f(q, \omega), \\ \delta\rho_f(q, \omega) &= \chi_f(q, \omega) g_{bf} \delta\rho_b(q, \omega), \end{aligned} \tag{8}$$

with retarded density-density response functions

$$\chi(q, \omega) = \frac{1}{\hbar} \sum_{n \neq 0} |\langle n|\hat{\delta}\rho^\dagger(q)|0\rangle|^2 \left[\frac{2\omega_{n0}}{(\omega + i\eta)^2 - \omega_{n0}^2}\right]. \tag{9}$$

The small imaginary shift $i\eta$ preserves causality and the ground state $|0\rangle$ represents the Fermi sea for the fermions

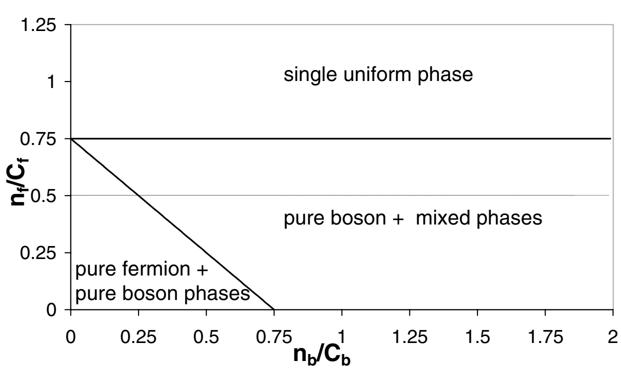

FIG. 1. Phase diagram for a mixture of bosons and fermions in one dimension. The thin line corresponds to the linear stability condition in Eq. (5).

and the condensate for bosons. In the Bogoliubov approximation for the bosons the response function as well as the quasiparticle spectrum in 1D have algebraic forms identical to those in 3D and are given by

$$\chi_b(q, \omega) = (n_b q^2)/\{m_b[\omega^2 - \omega_b^2(q)]\}, \tag{10}$$

with $\omega_b^2(q) = (\epsilon_q/\hbar)^2 + (v_b q)^2$, free quasiparticle energy $\epsilon_q = \hbar^2 q^2/2m_b$, and sound velocity $v_b = \sqrt{g_b n_b/m_b}$. The poles correspond to the energies of the collective modes, undamped in the Bogoliubov approximation. The response function for the fermions is that for free fermions, which has a form in 1D quite distinct from that in 3D:

$$\begin{aligned} \chi_f(q, \omega) &= \frac{m_f}{2\pi\hbar^2 q} \ln\left[\frac{(\omega + i\eta)^2 - \omega_-^2(q)}{(\omega + i\eta)^2 - \omega_+^2(q)}\right], \\ \hbar^2\omega_\pm^2(q) &= (\hbar^4/4m_f^2)[(k_f \pm q)^2 - k_f^2]^2. \end{aligned} \tag{11}$$

The calculation leading from Eq. (9) to Eq. (11) is analogous to that in 3D [18] with the Fermi sphere replaced by a "Fermi interval" $[-k_F, k_F]$. It is apparent that $\mathrm{Im}\chi_f(q, \omega) \neq 0$ only if $|\omega_-| \leq |\omega| \leq |\omega_+|$.

The low energy, long wavelength modes are of particular experimental interest, so we do a Taylor expansion of the expression (11) for χ_f for small values of q, but keeping the ratio of the energy transfer to momentum transfer ω/q constant. The result is quite interesting:

$$\chi_f(q, \omega) \simeq (n_f q^2)/\{m_f[(\omega + i\eta)^2 - \omega_f^2(q)]\}, \tag{12}$$

with $\omega_f^2(q) = (\epsilon_q/\hbar)^2 + (v_f q)^2$. If we replace the Bogoliubov sound velocity with the Fermi velocity $v_b \to v_f = \hbar k_f/m$, the real part of this limiting form is *identical* to the Bogoliubov response function, and the spectrum corresponding to their poles are identical in form. This equivalence of the bosonic and fermionic density fluctuations is a distinct property of 1D with no analog in higher dimensions. Such an equivalence is not surprising if one recalls the Luttinger liquid model of Haldane [19] where the low energy behavior of quantum fluids in 1D was shown to be independent of spin-statistics. However, it is important to note that the fermionic response function that we consider above is that for free fermions, while the Luttinger-Tomonaga [10] model assumes long range interactions among the fermions. The absence of interaction among fermions distinguishes the qualitative nature of the fermionic excitations from the Bogoliubov modes, despite the similarity in the low energy structure of the response functions. The fermionic excitations are elementary excitations, whereas the Bogoliubov modes are collective modes of the bosons for which the interactions play a central role.

The similarity of the response functions at low energies combined with the fact that homogeneous Bogoliubov modes have the same form in 1D and 3D allows us to directly apply the results obtained for spatially uniform binary mixtures of bosons in 3D to the BFM in 1D. As an example, we consider the normal modes of the BFM determined by the vanishing of the coefficient determinant of Eqs. (8), $1 - g_{bf}^2 \chi_b \chi_f = 0$, which leads to an expression for the normal mode velocities similar to that for binary mixtures of bosons in 3D [20]

$$v_{\pm}^2 = \frac{1}{2}\left[(v_b^2 + v_f^2) \pm \sqrt{(v_b^2 - v_f^2)^2 + 4g_{bf}^2 \frac{n_f n_b}{m_f m_b}}\right]. \quad (13)$$

Here we have used the linear dispersions $\omega_{b(f)}(q) \simeq v_{b(f)} q$ for long wavelength modes. Hydrodynamic equations for boson-fermion density fluctuations in the collisional regime also give a similar expression for sound velocities [12], but with the crucial difference that the fermion sound velocity in that case is that of ordinary sound, $v_f/\sqrt{3}$. In the long wavelength limit when $q \to 0$, v_-^2 may become negative, implying an imaginary mode frequency, a sign of a dynamic instability due to an exponentially growing mode. The condition for v_-^2 to be non-negative leads to exactly the condition for phase stability obtained earlier in Eq. (5) from energy considerations. This dynamic instability associated with long wavelength modes leads to phase segregation.

The frequencies corresponding to the original low energy Bogoliubov phonons shift due to the interaction with the fermions by $\omega_q - v_b q \simeq n_f n_b g_{bf}^2 q^2 / [m_b m_f (v_b^2 - v_f^2)]$; the shift is positive for $v_b > v_f$ and negative for $v_b < v_f$, similar to the behavior in 3D [13]. Such low energy modes in 3D undergo Landau damping if $v_b < v_f$. In the 1D mixture these modes are undamped for both $v_b > v_f$ and $v_b < v_f$ as the imaginary part of the fermionic response function, that leads to damping, vanishes due to the identity $\mathrm{Im}\{(\omega \pm i\eta)^{-1}\} = \mp \pi \delta(\omega)$. This is a consequence of the low energy similarity of the fermionic response with the Bogoliubov response in 1D.

When we consider modes with higher momenta where the exact fermion response function has to be used, there will be Landau damping even in 1D but only for mode frequencies in the range $|\omega_-| \leq |\omega| \leq |\omega_+|$; only in this range is the argument of the logarithm in Eq. (11) negative and yields an imaginary component. For weak interaction the damping rate for the normal modes can be calculated from their defining equation $g_{bf}^2 \chi_b \chi_f = 1$ to be $\gamma \sim m c_b g_{bf}^2 / 4\hbar^2 g_b$. The key distinction is that the rate is *independent* of the mode; in 3D the damping is linearly dependent on the momentum of the mode [13].

The exact fermionic response function (11) in 1D has several features distinct from 3D. For zero energy transfer $\chi_f(q, 0)$ has a logarithmic divergence at $q = 2k_f$ due to perfect nesting, whereas in 3D the derivative of the response function is divergent, which leads to Friedel oscillations. In 1D BFM, the divergence of response function leads to periodic density variations of the fermions of period $2k_f$ associated with the formation of coherent superposition of particle-hole pair states called the Peierls channel [21]. Because of the coupling with fermions, the bosons acquire a similar periodicity but out of phase with the fermion density modulation [22].

In conclusion, we have studied the phase stability of a boson-fermion mixture in 1D and demonstrated that phase separation would occur at low fermion densities, a behavior opposite to that in 3D. This means that phase separation effects may be studied at densities easier to achieve than in 3D. The regimes of coexistence of bosons and fermions in the same space are within the reach of experimental capabilities, and there is the exciting prospect of creating degenerate fermions in 1D. Also we have shown that the low energy density-density response of free fermions is identical in form to that of weakly interacting bosons; this means that binary mixtures will have similar normal modes regardless of whether the components are bosons or fermions. These modes are undamped at low momenta. Away from the low energy regime, the similarity of response functions does not hold.

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[1] M. H. Anderson *et al.*, Science **269**, 198 (1995); C. C. Bradley *et al.*, Phys. Rev. Lett. **75**, 1687 (1995); K. B. Davis *et al.*, Phys. Rev. Lett. **75**, 3969 (1995).
[2] B. DeMarco and D. S. Jin, Science **285**, 1703 (1999).
[3] A. G. Truscott *et al.*, Science **291**, 2570 (2001).
[4] F. Schreck *et al.*, Phys. Rev. Lett. **87**, 080403 (2001).
[5] W. Hänsel *et al.*, Nature (London) **413**, 498 (2001).
[6] H. Ott *et al.*, Phys. Rev. Lett. **87**, 230401 (2001).
[7] A. E. Leanhardt *et al.*, Phys. Rev. Lett. **89**, 040401 (2002).
[8] A. Görlitz *et al.*, Phys. Rev. Lett. **87**, 130402 (2001).
[9] M. Greiner *et al.*, Phys. Rev. Lett. **87**, 160405 (2001).
[10] J. M. Luttinger, J. Math. Phys. (N.Y.) **4**, 1154 (1963); S. Tomonaga, Prog. Theor. Phys. **5**, 544 (1950).
[11] L. Viverit, C. J. Pethick, and H. Smith, Phys. Rev. A **61**, 053605 (2000).
[12] A. Minguzzi and M. Tosi, Phys. Lett. A **268**, 142 (2000).
[13] S. K. Yip, Phys. Rev. A **64**, 023609 (2001).
[14] K. Molmer, Phys. Rev. Lett. **80**, 1804 (1998); M. J. Bijlsma, B. A. Heringa, and H. T. C. Stoof, Phys. Rev. A **61**, 053601 (2000); T. Miyakawa, T. Suzuki, and H. Yabu, *ibid.* **64**, 033611 (2001); A. P. Albus *et al.*, *ibid.* **65**, 053607 (2002); R. Roth, *ibid.* **66**, 013614 (2002); C. P. Search *et al.*, Phys. Rev. A **65**, 063615 (2002); L. Viverit and S. Giorgini, *ibid.* **66**, 063604 (2002).
[15] J. A. Sauer, M. D. Barrett, and M. S. Chapman, Phys. Rev. Lett. **87**, 270401 (2001).
[16] Kunal Das, Phys. Rev. A **66**, 053612 (2002).
[17] Z. Hadzibabic *et al.*, Phys. Rev. Lett. **88**, 160401 (2002).
[18] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).
[19] F. D. M. Haldane, Phys. Rev. Lett. **47**, 1840 (1981).
[20] P. Ao and S. T. Chui, J. Phys. B **33**, 535 (2000).
[21] George Grüner, *Density Waves in Solids* (Addison-Wesley, Reading, MA, 1994).
[22] T. Miyakawa *et al.*, in *Proceedings of the XVIII International Conference on Atomic Physics* (World Scientific, Singapore, 2002).