d-Dimensional Black Hole Entropy Spectrum from Quasinormal Modes

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Starting from recent observations about quasinormal modes, we use semiclassical arguments to derive the Bekenstein-Hawking entropy spectrum for d-dimensional spherically symmetric black holes. We find that, as first suggested by Bekenstein, the entropy spectrum is equally spaced: $S_{\rm BH} = k\ln(m_0)n$, where m_0 is a fixed integer that must be derived from the microscopic theory. As shown in O. Dreyer, gr-qc/0211076, 4D loop quantum gravity yields precisely such a spectrum with $m_0 = 3$ providing the Immirzi parameter is chosen appropriately. For d-dimensional black holes of radius $R_H(M)$, our analysis predicts the existence of a unique quasinormal mode frequency in the large damping limit $\omega^{(d)}(M) = \alpha^{(d)}c/R_H(M)$ with coefficient $\alpha^{(d)} = \frac{(d-3)}{4\pi}\ln(m_0)$, where m_0 is an integer.

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Introduction.—There is a fairly wide consensus that the mass/entropy spectrum of black holes is discrete rather than continuous. Two methods currently exist for deriving this spectrum. The more fundamental and difficult one is to count black hole states of a fixed energy within some microscopic theory of quantum gravity in order to compute the statistical mechanical entropy. This has been done with some success in string theory [1] (for extremal and near extremal black holes) and in loop quantum gravity [2]. The second method consists of quantizing (usually within a mini-super-space approach) certain dynamical modes of the classical theory and evaluating the quantum spectrum of the Bekenstein-Hawking entropy, $S_{\rm BH}$, which by definition is one-quarter of the horizon area expressed in Planck units.

Bekenstein was the first to claim on heuristic grounds that black hole entropy is an adiabatic invariant with an equally spaced quantum spectrum [3]. Bekenstein and Mukhanov [4] argued that in order for this entropy to have a statistical mechanical interpretation, it must correspond to the logarithm of an integer: $S_{\rm BH} = k\ln(\Omega)$, with $\Omega = 2^n$ being the most natural choice. There have been many attempts to derive this entropy spectrum directly from the dynamical modes of the classical theory [5–10]. Discrete spectra arise in quantum mechanics in the presence of a periodicity in the classical system which in turn leads to the existence of an adiabatic invariant or action variable. Bohr-Sommerfeld quantization implies that this adiabatic invariant has an equally spaced spectrum in the semiclassical limit. From this viewpoint, the main problem of black hole quantum mechanics has been to correctly identify the physically relevant period or vibrational frequency. For example, in [7], the period was taken to be proportional to the inverse Hawking temperature motivated by the corresponding periodicity in the Euclidean form of the black hole partition function [9]. As shown generically in [10] this leads to the result that the Bekenstein-Hawking entropy is indeed an adiabatic invariant with an equally spaced quantum spectrum.

To date there has been very little known about the direct physical connection between the classical dynamical quantities that give rise to the Bekenstein-Hawking entropy and the corresponding microscopic degrees of freedom of the quantum black hole. A potentially important step in this direction was made by Hod [11] in the context of the quasinormal modes of the classical theory [12]. In particular, he assumed an equally spaced area spectrum and used the existence of a unique quasinormal mode frequency in the large damping limit to uniquely fix the spacing. Remarkably, Hod found that the resulting spacing was such as to allow a statistical mechanical interpretation for the eigenvalues of the Bekenstein-Hawking entropy. Dreyer [13] also used the large damping quasinormal mode frequency to fix the value of the Immirzi parameter, γ, in loop quantum gravity. This value of γ turned out to be precisely the one required to make the loop quantum gravity entropy prediction coincide with the classical Bekenstein-Hawking entropy.

In the present Letter we first use these observations about the quasinormal mode frequency to derive the general form in the semiclassical limit of the Bekenstein-Hawking entropy spectrum for d-dimensional spherically symmetric black holes. Note that [11] assumed an equally spaced area spectrum, whereas we show how it arises as a consequence of taking the large damping quasinormal mode frequency seriously in the context of black hole dynamics. Moreover, we argue that this analysis allows us to predict the d-dimensional large damping quasinormal mode frequency up to a single arbitary integer. If this prediction is confirmed by numerical calculations, then the entropy spectrum could be used as a test for the viability of any proposed microscopic theory of quantum gravity.

We first derive the entropy spectrum for 4D black holes in a way that is not tied to any particular microscopic theory of quantum gravity. Next we examine the consequences of this interpretation for higher dimensional, spherically symmetric black holes. The final section contains conclusions.

4D black holes.—We start from the observation that for a Schwarzschild black hole of mass M and radius $R_H(M)$, the real part of the quasinormal mode frequency approaches a fixed nonzero value in the large damping limit. This value is

$$\omega_{\rm QNM}(M) = \alpha^{(4)}\left(\frac{c}{R_H(M)}\right). \tag{1}$$

The functional form (c/R_H) is expected from dimensional arguments: it is just the inverse of the horizon light transit time. The coefficient $\alpha^{(4)}$ has been determined numerically [12,14]:

$$\alpha^{(4)} = 0.043\,712\,35 \approx \frac{\ln(3)}{4\pi}. \tag{2}$$

Hod was the first to note and make use of the fact that the numerical value of $\alpha^{(4)}$ agrees to the given accuracy with the analytic expression given on the right-hand side of (2).

Following [11,13], we assume that this classical frequency plays an important role in the dynamics of the black hole and is relevant to its quantum properties. In particular, we consider $\omega_{\rm QNM}(M)$ to be a fundamental vibrational frequency for a black hole of energy $E = Mc^2$. Given a system with energy E and vibrational frequency $\omega(E)$ it is a straightforward exercise to show that the quantity

$$I = \int \frac{dE}{\omega(E)} \tag{3}$$

is an adiabatic invariant, which via Bohr-Sommerfeld quantization has an equally spaced spectrum in the semiclassical (large n) limit

$$I \approx n\hbar. \tag{4}$$

By taking $\omega_{\rm QNM}$ seriously in this context, we are led to the adiabatic invariant

$$I(E) = \frac{c^3}{2G\alpha^{(4)}}\int dEE = \frac{\hbar}{4\pi\alpha^{(4)}}S_{\rm BH}(E) + \text{const}, \tag{5}$$

where we have used the fact that $R_H = 2GM/c^2 = 2GE/c^4$ and the definition of the Bekenstein-Hawking entropy $S_{\rm BH} = \pi R_H^2/(\hbar G)$. Bohr-Sommerfeld quantization then implies that the entropy spectrum is equally spaced, with coefficient determined by $\alpha^{(4)}$:

$$S_{\rm BH} = 4\pi\alpha^{(4)} n = \ln(3)n. \tag{6}$$

The miracle, first observed in [11], is that the numerical value of $\alpha^{(4)}$ allows a statistical mechanical interpretation for this entropy. That is, $S_{\rm BH}$ is the logarithm of an integer, which can be interpreted as the degeneracy of quantum states:

$$\Omega(E) = \exp(S_{\rm BH}) = 3^n. \tag{7}$$

It is important to note that this prediction for the degeneracy of states comes directly from the postulated physical interpretation of the quasinormal mode $\omega_{\rm QNM}$. There are no free parameters once α is "measured" numerically. It therefore provides, in principle, a constraint that any viable quantum theory of gravity must satisfy. Equation (7) is similar in form to the degeneracy $\Omega = 2^n$ advocated by Bekenstein and Mukhanov [4]. Moreover, with suitable assumptions about the gauge group, loop quantum gravity is able to predict precisely this degeneracy of states [13].

D-dimensional Schwarzschild solution.—We now extend the above considerations to spherically symmetric black holes in d dimensions (the so-called Schwarzschild-Tangherlini black holes [15]). The solution for the metric, expressed in terms of the Arnowitt-Deser-Misner energy E is

$$ds^2 = -\left(1 - \frac{16\pi G^{(d)}E}{(d-2)\Gamma^{(d-2)}c^4 r^{d-3}}\right) + \left(1 - \frac{16\pi G^{(d)}E}{(d-2)\Gamma^{(d-2)}c^4 r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega^{(d-2)}, \tag{8}$$

where $G^{(d)}$ is the d-dimensional Newton constant, $d\Omega^{(d-2)}$ is the volume element on a unit $(d-2)$ sphere, and $\Gamma^{(d-2)}$ is the volume of the unit $(d-2)$ sphere. The corresponding horizon radius is

$$R_H(E) = \left(\frac{16\pi G^{(d)}E}{(d-2)\Gamma^{(d-2)}c^4}\right)^{1/(d-3)} \tag{9}$$

and the associated Bekenstein-Hawking entropy is

$$S_{\rm BH}(E) = \frac{1}{4}\left(\frac{c^3\Gamma^{(d-2)}R_H^{d-2}(E)}{\hbar G^{(d)}}\right). \tag{10}$$

By applying the first law of thermodynamics, $dS = dE/T_{\rm BH}$, one can deduce the general expression for the Bekenstein-Hawking temperature:

$$T_{\rm BH}(E) = \frac{\hbar c(d-3)}{4\pi R_H(E)}. \tag{11}$$

We now assume that, as in 4D, there is a unique quasinormal mode frequency in the large damping limit whose value on dimensional grounds takes the general form

$$\omega_{\rm QNM}(E) = \alpha^{(d)}\frac{c}{R_H(E)}. \tag{12}$$

Note that the existence of such a unique limit in dimensions other than 4 has not been established, so that the coefficient is as yet unknown. The same semiclassical argument as in the previous section suggests the existence of an adiabatic invariant associated with d-dimensional black holes:

$$I^{(d)}(E) = \int \frac{dE}{\omega_{\rm QNM}(E)} = \frac{\hbar(d-3)}{4\pi\alpha^{(d)}} \int \frac{dE}{T_{\rm BH}(E)} = \frac{\hbar(d-3)}{4\pi\alpha^{(d)}} S_{\rm BH}(E), \tag{13}$$

where we have used (11) and (12) to replace ω in the denominator by the Bekenstein-Hawking temperature, and the first law to arrive at the final expression in terms of the entropy. Bohr-Sommerfeld quantization of the adiabatic invariant $I = n\hbar$ thus gives rise to an equally spaced entropy spectrum in the semiclassical limit:

$$S_{\rm BH}(E) = \frac{4\pi\alpha}{(d-3)} n. \tag{14}$$

It is important to remember that it is not merely a numerical coincidence that the Bekenstein-Hawking entropy emerges as the adiabatic invariant associated with the quasinormal mode vibrational frequency. Dimensional arguments suggest that for Schwarzschild black holes the frequency is proportional to c/R_H, which in turn is proportional to the Hawking temperature of the black hole (in any spacetime dimension). Thus the adiabatic invariant I is generically

$$I \propto \int \frac{dE}{T_{\rm BH}}. \tag{15}$$

As first argued in [10], it is therefore a direct consequence of the first law of thermodynamics that the entropy is an adiabatic invariant with an equally spaced quantum spectrum. An interesting question is whether this relationship also holds for charged or rotating black holes.

If one makes the additional assumption that the Bekenstein-Hawking entropy is actually the statistical mechanical entropy of the black hole states associated with the microscopic quantum gravity theory, then the quantity $\Omega(E) = \exp(S_{\rm BH}) =: (m_0)^n$ must be an integer. This in turn suggests the following form for the coefficient of the quasinormal mode frequency:

$$\alpha^{(d)} = \frac{(d-3)}{4\pi} \ln(m_0), \tag{16}$$

where m_0 is an unspecified integer that depends on the microscopic theory.

Conclusions.—We have argued as in [11,13] that the unique large damping quasinormal mode of spherically symmetric black holes is a fundamental frequency associated with the dynamical system that can be used to gain semiclassical information about the entropy spectrum of generic black holes. This generically leads directly to the result that Bekenstein-Hawking entropy is an adiabatic invariant with an equally spaced quantum spectrum. Moreover, the spacing is uniquely fixed by the value of the large damping quasinormal frequency. Based on Hod's observations for 4-dimensional Schwarzschild black holes, we conjecture that the values of the higher dimensional large damping quasinormal modes are such as to give a Bekenstein-Hawking entropy spectrum of the form

$$S_{\rm BH} = \ln(m_0) n, \tag{17}$$

which has, in principle, a statistical mechanical interpretation. If correct, this implies that the quasinormal frequencies, which are purely classical quantities, are providing information about the microscopic degrees of freedom associated with the underlying theory of quantum gravity. This, in principle, provides a potentially useful constraint on proposed microscopic theories: any viable theory of quantum gravity would have to contain black hole states in its spectrum with entropy as given by (17). Dreyer's work [13] shows that loop quantum gravity passes this test, providing the Immirzi parameter and gauge group are chosen judiciously.

The purpose of the present Letter has been not only to put forward the above conjectures, but more importantly, to make a concrete prediction with which they can be tested. If the coefficients, $\alpha^{(d)}$, of the large damping quasinormal modes in higher dimensions do not have the form given in (16), then Hod's results are special to four dimensions and do not have any general significance. If they do take the right form, then it would provide strong evidence for a deep link between quasinormal frequencies and black hole entropy.

Finally, we end with a speculation. One calculation that directly relates the classical dynamics of black holes to a statistical mechanical interpretation of the Bekenstein-Hawking entropy is due to Carlip [16]. In this approach, the huge black hole entropy is associated with a degeneracy of states arising from the existence of an asymptotic conformal symmetry at the horizon. The present work seems to suggest a possible connection classically between the unique quasinormal mode frequency in the large damping limit and this asymptotic conformal symmetry. In particular, the uniqueness of the quasinormal mode frequencies in the large damping limit may somehow be related to the conformal symmetry near the horizon. The fact that these frequencies are directly connected to the statistical mechanical entropy adds weight to this conjecture.

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Note added.—After this Letter was submitted for publication the precise forms for the large damping quasinormal modes were calculated analytically for 4-dimensional [17] and d-dimensional [18] Schwarzschild black holes. The results agree with (2) and (16).

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