Low-Dimensional Models for Vertically Falling Viscous Films

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Long wave evolution on free falling viscous films is described using a new evolution equation. The scaling proposed here brings in the viscous and pressure correction terms that are missing in the existing long-wave equations. Small amplitude expansion of the equation gives a dissipative form of the Kuromoto-Sivashinsky equation. Improved accuracy of the new equation over existing equations is demonstrated by comparison of neutral curves with Orr-Sommerfeld equations and experimental data.

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Surface waves on a vertically falling film are known to exhibit complex spatial and temporal patterns and have been a subject of interest for the last 50 years [1,2]. Studying the behavior of these waves with the governing Navier-Stokes (NS) equations is a complicated and computationally expensive task because of the nonlinearities in the equations and boundary conditions, and the presence of a moving interface. Alternatives to the NS equations for describing these waves are low-dimensional models derived from the NS under certain assumptions.

Single evolution equations form a category of low-dimensional models where the dynamics of the film is quantified with a partial differential equation in x and t for scaled film thickness $h\ (x, t)$. A well-known example of single evolution equation is the long-wave (LW) equation [3] derived by Benney (1966),

$$\frac{\partial h}{\partial t} + 3h^2 h_x + \frac{\partial}{\partial x}\left[\frac{3}{10}\mathrm{Re}h^6 h_x + \frac{\mathrm{ReWe}}{12}h^3 h_{xxx}\right] = 0. \quad (1)$$

Here, Re is the Reynolds number and We is the Weber number. This equation is obtained by a perturbation expansion on the NS in terms of wave number α, with the assumptions that Re and α^2We are of $O(1)$. A weakly nonlinear truncation of this equation gives the well-known Kuromoto-Sivashinsky (KS) equation.

It is now recognized that the above LW equation and extensions of it to higher orders in α, cannot describe the wave amplitudes observed in experiments [4–7]. The reason was identified to be the omission of pressure correction and viscous dissipation terms in these equations. Though higher order expansions [8,9] include these missing terms, comparison of linear stability results to that of Orr-Sommerfeld (OS) equations shows that these expansions also diverge in the linear limit. In this contribution, we present a new scaling and use it to develop a single evolution equation that can describe wave evolution on viscous films both qualitatively and quantitatively.

The two-dimensional NS equations for film flow down a vertical wall can be found in [4,7]. One of the solutions to these equations is the Nusselt's flat film solution obtained by balancing the viscous and gravitational forces. The velocity profile for this case is parabolic. The equations are made dimensionless by choosing Nusselt's average velocity u_N and film thickness h_N as the characteristic velocity and length scales. Time and pressure are scaled by h_N/u_N and $\mu u_N/4h_N$, respectively. The resulting system of equations has two independent dimensionless groups, the Kapitza number ($\mathrm{Ka} = \sigma\rho^{1/3}/g^{1/3}\mu^{4/3}$) and the Reynolds number ($\mathrm{Re} = 4u_N h_N/\nu = gh_N^3/3\nu^2$), where σ is the surface tension, ρ is the density, μ is the viscosity, and ν is the kinematic viscosity. The Kapitza number is a function of fluid physical properties and hence constant for a given fluid. A combination of Kapitza and Reynolds number gives the Weber number, defined as $\mathrm{We} = \sigma/(\rho u_N^2 h_N) = 3^{1/3}4^{5/3}\mathrm{Ka}/\mathrm{Re}^{5/3}$.

Traditionally, the Navier-Stokes equations are reduced to a single evolution equation using a perturbation expansion in α along with the assumptions $\mathrm{Re} \sim O(1)$ and $\alpha^2\mathrm{We} \sim O(1)$ (long-wave scaling). The wave number α is introduced into the dimensionless equations by scaling the length in the flow direction with an unknown wavelength $\lambda/2\pi$. The wave number $\alpha(= 2\pi h_N/\lambda)$ is assumed to be small. At the lowest order $O(\alpha^o)$ in the expansion, viscous and gravitational forces balance each other. The evolution equation at this order is given by

$$h_t + 3h^2 h_x = 0. \quad (2)$$

This equation describes the behavior of waves with infinitely large wavelengths. For describing finite wavelengths, the expansion should be carried to higher orders. At the next order $O(\alpha^1)$, inertial and capillary terms correct the lowest order equation as given by the LW equation (1). It should be noticed here that, irrespective of the magnitude of viscous, inertial, and capillary terms, the correction to the lowest order solution in the long-wave scaling always comes from the inertial and capillary terms while viscous terms are pushed to higher orders. Comparison of the neutral stability curves of this equation (1) to the OS equations shows that the curves diverge well within the region in which it is expected to be valid [4]. The equation also predicts that all the linearly unstable waves travel with the same dimensionless velocity (Ce_r) of 3. Thus, the equation does not show dispersion of waves due to missing viscous terms. To capture these viscous terms, Gjevik [8] and Nakaya [9] carried the expansion to higher orders. However, these higher order expansions also diverge from the OS results and hence do not retain the qualitative behavior of the waves [4].

The Reynolds number is normally used to distinguish between viscous and inertia-dominated regimes. For falling films, since the film thickness and velocity are not independent of each other, the magnitude of Reynolds number is not sufficient to distinguish between viscous and inertia-dominated regimes. For example, consider water and 95% glycerin solution at the same Reynolds number Re = 1.0. The Kapitza numbers for these two fluids are 3371 and 0.24, respectively. The Weber number can be calculated from Ka and Re. For water the Weber number is 48 980 while for glycerin it is 3.48. At this Reynolds number the thickness of water film is 0.0425 mm while that of glycerin is 4.25 mm. At the same Reynolds number Re = 1, film thickness of glycerin is 100 times larger than that of water. While viscous forces are dominant in the thin water film, both inertial and viscous forces are expected to be important in the thick glycerin film. Thus, the magnitude of Reynolds number alone cannot distinguish between viscous and inertia-dominated regimes. However, the Weber number is large for the water film and is of order unity for glycerin. Unlike the Reynolds number, a large Weber number implies viscous dominated regime and a small Weber number represents the inertia-dominated regime. Thus, the corrections to the lowest order terms should depend on the Weber number, not on the Reynolds number.

We replace the assumption $\mathrm{Re} \sim O(1)$ in the long-wave scaling with $\mathrm{Ka} \sim O(1)$ and retain the large Weber number $\alpha^2\mathrm{We} \sim O(1)$ assumption. Rearranging the relation between We, Re, and Ka we get $\mathrm{Re} = (3^{1/3}4^{5/3}\mathrm{Ka})^{3/5} \times (1/\mathrm{We})^{3/5} = (3^{1/3}4^{5/3}\mathrm{Ka}/W)^{3/5}\alpha^{6/5} = \beta\alpha^{6/5}$, where $\alpha^2\mathrm{We} = \mathrm{W} \sim O(1)$ and β is an $O(1)$ parameter. This new scaling suggests that viscous terms must be given more importance than inertial terms in the region where $\alpha \ll 1$ or ($\mathrm{We} \gg 1$). Inertial effects are dominant in the region where $\mathrm{We} \ll 1$ or $\alpha \gg 1$. Viscous and inertial effects are of equal order when $\mathrm{We} \sim O(1)$ or $\alpha \sim O(1)$. The large Weber number limit where viscous and/or capillary effects are strong is called the *viscocapillary regime*.

In terms of Reynolds number, the viscocapillary regime corresponds to $0 < \mathrm{Re} \ll 3^{1/5}4\mathrm{Ka}^{3/5} (\approx 5\mathrm{Ka}^{3/5})$. For water, it is given by $0 < \mathrm{Re} \ll 654$ while for 95% glycerin solution it is $0 < \mathrm{Re} \ll 2.0$. In the example discussed before, Re = 1 for water is well within the viscocapillary regime, but for glycerin Re = 1 lies on the border where viscous and inertial effects are significant. For $\mathrm{Re} \gg \mathrm{Ka}^{3/5}$ inertial effects become important. Thus, for small Kapitza numbers, inertial effects can become significant at low Reynolds numbers.

The advantages of the new scaling can be seen from the neutral stability curves of the OS equations. Figure 1 shows the neutral stability curves computed from the OS equations as a function of Reynolds number for different Kapitza numbers (different fluids). The Kapitza number is varied over a wide range of 1 to 3371. When the same curves are plotted again as a function of Weber number (Fig. 2) they collapse onto a single curve in the limit $1/\mathrm{We} \to 0$ and are close to each other until $\mathrm{We} \sim O(1)$. However, when plotted as a function of Reynolds number (Fig. 1), the curves diverge from each other at the origin. Another important observation that can be made from Fig. 2 is that, irrespective of the Kapitza number, the critical wave number $\alpha_c < 1$ as long as the Weber number is large, which shows that the long-wave assumption is valid for $\mathrm{We} \gg 1$.

Introducing the new scaling $\mathrm{Ka} \sim O(1)$ and $\alpha^2\mathrm{We} \sim O(1)$ into the dimensionless equations, we obtain at the zeroth order $O(\alpha^o)$, $h_t + 3h^2h_x = 0$. This equation is the same as obtained using the long-wave scaling. However, at the next order, pressure and viscous dissipation terms correct the lowest order solution, unlike the long-wave scaling where the correction comes from inertial terms. The equation at $O(\alpha^2)$ is given by

$$h_t + 3h^2h_x + \frac{\partial}{\partial x}[3h^4h_{xx} + 7h^3h_x^2] = 0. \qquad (3)$$

Because of the way the perturbation parameter α appears in the dimensionless equations, the evolution equation is not corrected at $O(\alpha^1)$. Inertial and capillary terms enter the evolution equation at the next order $O(\alpha^{1\,1/5})$. The equation at this order is given by

$$h_t + 3h^2h_x + \frac{\partial}{\partial x}[3h^4h_{xx} + 7h^3h_x^2] + \frac{\partial}{\partial x}\left[-\frac{5}{32}\mathrm{Re}h^4h_t - \frac{27}{160}\mathrm{Re}h^6h_x + \frac{\mathrm{ReWe}}{12}h^3h_{xxx}\right] = 0 \qquad (4)$$

Since this equation includes all three viscous, inertial, and capillary terms, we truncate the expansion at this

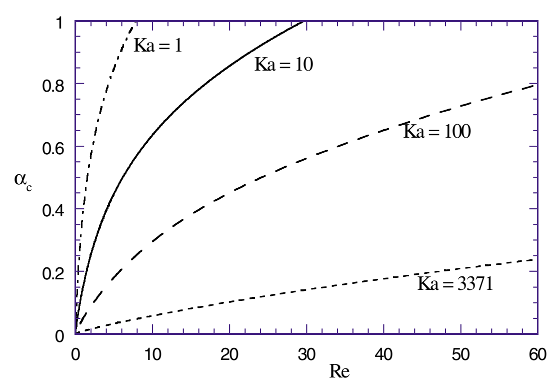

FIG. 1 (color online). Neutral stability curves obtained from Orr-Sommerfeld equations for different fluids.

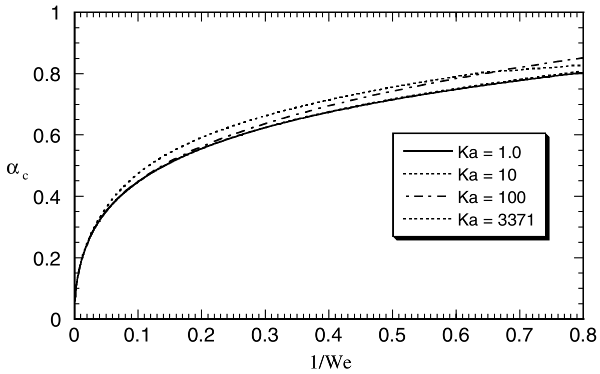

FIG. 2. Neutral stability curves in Fig. 1 are plotted with the reciprocal of Weber number.

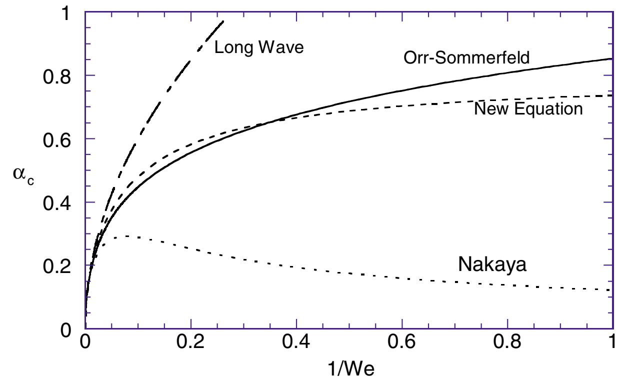

FIG. 3 (color online). Neutral stability curves for Ka = 10.

order. Comparing Eq. (4) to the LW Eq. (1), it can be clearly seen that the new equation includes viscous dissipation and pressure correction terms. Moreover, in the LW Eq. (1) the time derivative in the term $\frac{5}{32}\mathrm{Re}h^4h_t$ is replaced by its zeroth order approximation $h_t = -3h^2h_x$. We observe that this approximation, commonly made while deriving single evolution equations yields nonphysical wave celerities and leads to poor qualitative and quantitative agreement with OS results and experimental data.

Standard temporal linear stability analysis of Eq. (4) around the flat film solution $h = 1$ yields

$$\mathrm{Ce}_r = \frac{3 + (\frac{135}{5120}\mathrm{Re}^2 - 3)\alpha^2 + \frac{5}{384}\mathrm{Re}^2\mathrm{We}\alpha^4}{1 + \frac{25}{1024}\mathrm{Re}^2\alpha^2}, \quad (5)$$

$$\mathrm{Ce}_i = \frac{\frac{3}{10}\mathrm{Re}\alpha - (\frac{15}{32} + \frac{\mathrm{We}}{12})\mathrm{Re}\alpha^3}{1 + \frac{25}{1024}\mathrm{Re}^2\alpha^2}, \quad (6)$$

where Ce_r is the celerity of the waves and $\alpha\mathrm{Ce}_i$ gives the growth rate of the waves. A similar analysis of the LW Eq. (1) gives

$$\mathrm{Ce}_r = 3; \quad \mathrm{Ce}_i = \frac{3}{10}\mathrm{Re}\alpha - \frac{\mathrm{ReWe}}{12}\alpha^3. \quad (7)$$

Comparing the results of the long-wave and the new equation, it can be seen that the new equation includes dispersion of the waves and relates the celerity of the waves to the system parameters Re and We. Neutral stability curves of different single evolution equations are shown in Fig. 3. From the figure it can be seen that the LW and Nakaya's (extension of the LW to higher orders) equation diverge from the Orr-Sommerfeld predictions, while the new equation follows the OS very closely in the viscocapillary regime and preserves the qualitative trend till Weber numbers of order unity.

The predictions of the new Eq. (4) in the nonlinear regime are also determined and compared to experimental data. The equation is analyzed in a steady traveling wave coordinate ($z = x - \mathrm{Ce}t$, $\frac{\partial}{\partial x} = \frac{\partial}{\partial z}$, $\frac{\partial}{\partial t} = -\mathrm{Ce}\frac{\partial}{\partial z}$. Ce is the celerity of the wave). In this coordinate, Eq. (4) can be written as a set of three ordinary differential equations. The dimensionless parameters are Ka, We, and Ce. Our analysis shows that the most complex behavior exhibited by the waves in the viscocapillary regime at low Kapitza numbers is either periodic or period two solutions. Figures 4 and 5 show the wave traces obtained from numerical simulation and experimental data for Ka = 5.9 and Re = 2.0. The experimental data are obtained from [4,5]. (Spectral analysis of the experimental trace shows a single dominant frequency.) For higher values of Kapitza number, the wave structure becomes more complex. In the asymptotic limit of $\mathrm{Ce} \rightarrow 3$ and $1/\mathrm{We} \rightarrow 0$ (long-wave limit), the system of ODE's has a double zero eigenvalue. We use the center manifold reduction and Melnikov's perturbation analysis [4] to develop analytical correlations for maximum wave amplitude and celerity in this limit:

$$h_{\max} - 1 = (3 - \mathrm{Ce})/6 = 63/(25\ \mathrm{We}). \quad (8)$$

As expected, these relations depend only on the Weber number and not on the Reynolds number. Figure 6 shows the comparison of experimentally obtained maximum wave amplitudes (reported in [4]) to the double zero scaling [Eq. (8)] obtained from the model. The maximum wave amplitudes predicted by Eq. (4) are in good agreement with the experimental data.

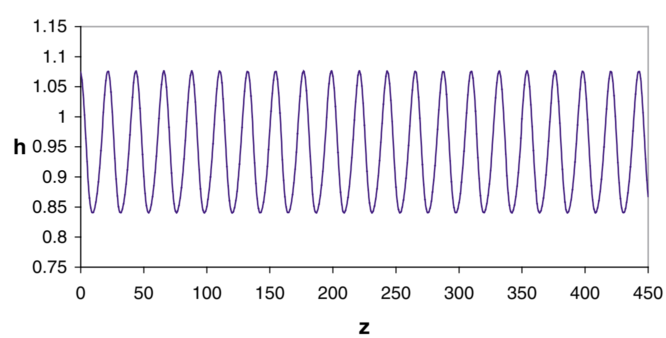

FIG. 4 (color online). Numerical wave trace for Ka = 5.9 and Re = 2.0.

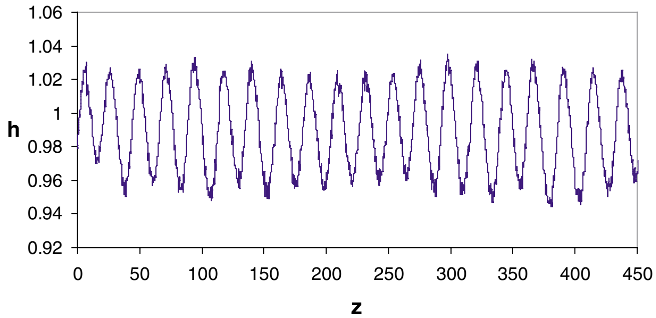

FIG. 5 (color online). Experimental wave trace for Ka = 5.9 and Re = 2.0.

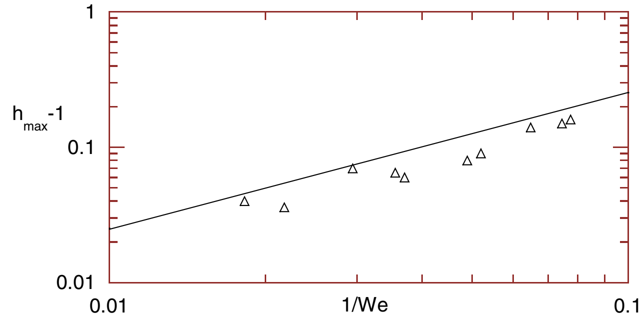

FIG. 6 (color online). Triangles denote experimental data and the straight line shows predictions of Eq. (8).

The weakly nonlinear form of Eq. (4) can be obtained by expanding h in terms of v as, $h = 1 + \alpha^2 v$, and retaining terms up to O($\alpha^{1\,1/5}$). The resulting equation in a coordinate system traveling with velocity three after scaling v, x, and t is given by

$$v_t + v v_x + \delta v_{xxx} - \gamma v_{tx} + v_{xx} + v_{xxxx} = 0, \qquad (9)$$

where $\delta = 3.807\ \mathrm{We}^{0.1}/\mathrm{Ka}^{0.6}$, $\gamma = 1.477\ \mathrm{Ka}^{0.6}/\mathrm{We}^{1.1}$. The above equation reduces to the widely studied KS equation for $\delta = \gamma = 0$. The KS equation omits the dissipative viscous term v_{xxx} and replaces the mixed derivative term v_{xt} by $-3v_{xx}$. For $\gamma = 0$, Eq. (9) reduces to the Kawahara [10] equation, widely used for description of dissipative systems. Equation (9) suggests that for physical systems where dissipative effects are not negligible, the KS and Kawahara equations need to be corrected with *two* parameters to retain the structure of solutions exhibited by the complete NS equations. Several examples of such systems can be found in [10]. Based on the results presented in Figs. 1 and 6, we believe that interfacial instabilities with significant dissipative effects should be described using equations similar to (4) and (9).

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