Dense, Rapid Flows of Inelastic Grains under Gravity

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The standard hydrodynamic description does not apply to the rapid flow regime of inelastic grains in the dense limit. Emphasizing the role of inelastic loss and collapse, we propose a new approach relying on a nonlocal dissipation scheme. Our model succeeds in accounting qualitatively and quantitatively for the linear profile of velocity found in experiments on dense gravity-driven flows.

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During the past couple of decades, a number of attempts have been made to adapt the classical kinetic theory of hard-sphere gases to systems of macroscopic, inelastic grains [1–4], and many experiments have been carried out in parallel in order to gain insight into the rheological properties of grain flows. Investigations have been conducted in various setups, such as inclined channels [5–7], flumes [8,9], annular shear cells [10–12], or rotating drums [13,14]. So far, there is a reasonable qualitative agreement between the predictions of kinetic theories and results of experiments conducted on dilute or moderately dense granular media; that is, the relation $\gamma \propto (1 - e^2)^{1/2} d^{-1} T^{1/2} F(\nu)$ is obeyed locally (γ is the shear rate, e is the elastic restitution coefficient, d the grain diameter, T the "granular temperature" [15], ν is the solid fraction, and $F(\nu)$ results from density correlations in the collision integral). However, experiments conducted on dense rapid granular flows lead to drastically different results, results which appear to lie beyond the domain of validity of standard hydrodynamic descriptions. Experimental results from inclined channel geometries [5,6,8] as well as from rotating drums [13,14,16,17] in two or three dimensions compare well, in that they exhibit the following properties. First the solid fraction appears as nearly constant in the flowing layer, with value $\nu_m \approx 0.8$ (in two dimensions) or $\nu_m \approx 0.64$ (in three dimensions) corresponding to the random close packing (except for the very upper region, owing to the unevenness of the free surface). Second, the shear rate γ is found to be *independent* of the depth, i.e., the velocity profile is linear (Figs. 1 and 2), and its order of magnitude is given by $\sqrt{g/d}$ (where g is the gravity constant and d is the grain diameter). Interestingly, the rheological behavior is found to be *insensitive* to the value of the restitution coefficient e [16]. As for the instantaneous velocity fluctuations in densely packed materials, it is important to realize that reliable data are very difficult to obtain experimentally. Real experiments can access grain trajectories only to within the experimental time resolution, and the ostensible velocity fluctuations result more from the sliding of grains over adjacent corrugated layers of particles than from the ballistic flight of grains punctuated by distinct collisions. Sampling of grain displacements over a very short time compared to γ^{-1} indicates a constant value for the velocity fluctuations through the flow, except for a thin transitional layer (1–2 grain diameters) localized at the bottom interface.

The above observations cannot be accounted for by kinetic theory. Indeed, introducing a gravity force into the momentum balance leads to the relation $\partial \sigma_{xz}/\partial z = \rho g \sin\theta$ in the steady regime (where x is the direction of the flow, and z is oriented downwards, normal to the flow). Since $\rho(z) \approx$ const, a straightforward integration of the momentum equation yields $\sigma_{xz} \approx \rho g z \sin\theta$, which simply expresses the linear dependence of the shear stress on depth. According to kinetic theory, a constant-density, isothermal gravity-driven flow of this type should exhibit Bagnold's quadratic dependence of the shear stress on shear rate [18], since both the collisional momentum exchange and the collision rate are proportional to γ. One therefore should obtain $v_x \propto (\rho g h^3 \sin\theta)^{1/2} \times [1 - (z/h)^{3/2}]$, and the 3/2-power law with respect to

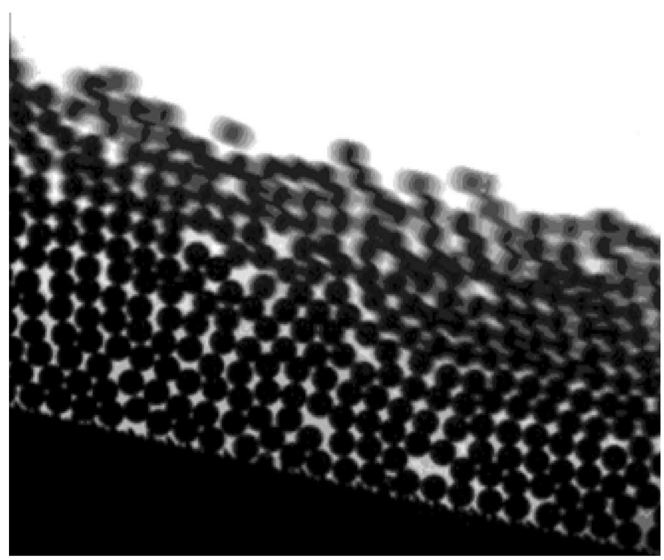

FIG. 1. Flow of a collection of monodisperse aluminum spheres (restitution coefficient $e = 0.6$, flow rate $\approx$1100 grains/sec, exposure time of the photograph: 1/125 sec).

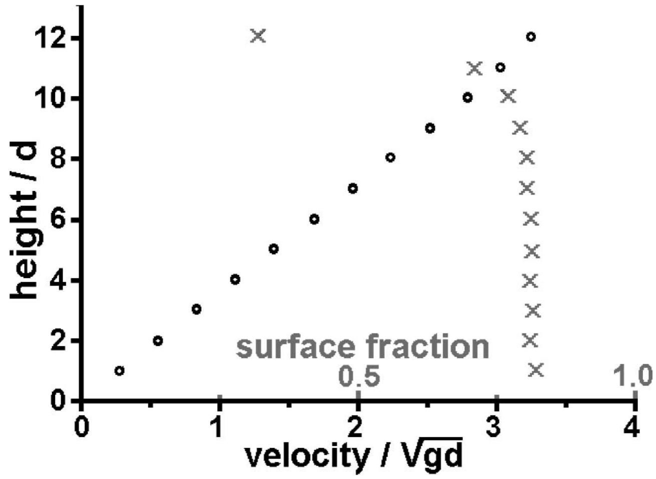

FIG. 2. Flow of aluminum grains, flow rate ≈ 1100 grains/sec, averaged over 100 samplings. (○) Adimensioned velocity profile; (×) surface fraction.

the depth disagrees with the linear velocity profiles found experimentally in dense media. On the other hand, as mentioned above, kinetic theory agrees fairly well with experiments on dilute media. We propose here to address the issue of the anomalous experimental results and to point out the reason for the inability of kinetic theory to capture the behavior of rapid, dense grain flows. We propose then a theoretical framework more suited to the rheology of rapid grain flows in the dense limit. Finally we address the role of intergranular friction.

In inclined channels a striking change of behavior can be noticed according to the value of the elastic restitution coefficient and to the depth of the flow [5,6]. For e close to 1 and/or for shallow beds, isolated grains undergo ballistic flight between successive binary collisions; this corresponds to the dilute collisional regime described by kinetic theories. On the other hand, for a small value of the restitution coefficient, or for deep beds, grains are densely packed. The latter situation corresponds to the regime characterized by a linear profile of velocity, a close-packed density, and constant velocity fluctuations over the depth in the flowing layer. We interpret this change from a dilute to a dense regime as follows. As first demonstrated by Bernu and Mazighi in the one-dimensional case [19], both momentum and energy are fully damped after $N \approx \pi/(1-e)$ impacts in the case of a collision wave propagating along a one-dimensional array. For ten grain diameters (which is the typical thickness of granular layers studied in laboratory experiments), this one-dimensional picture provides an estimate of $e \approx 0.7$ for the occurrence of the inelastic collapse of the flowing layer. The inelastic collapse phenomenon has been generalized by McNamara and Young [20] to two dimensions and the collapse criterion reads $N = \ln[(1-e)/4]/\ln[(1+e)/2]$. One gets thus $e \approx 0.6$ for $N = 10$.

In view of the above considerations, it is improper to consider isolated binary collisions in collapsed systems, because a nearly infinite number of impacts occurs in a finite time, completely attenuating both momentum and kinetic energy throughout the bulk [19–21]. The associated damping time is very short compared to the characteristic time γ^{-1} of the shearing. As a result of this *nonlocal* dissipation, the relative kinetic energy and momentum of two colliding particles fall very rapidly to zero. Consequently, collisions cannot be modeled with the original restitution coefficient e, since they are virtually completely inelastic. This is why the observed rheology appears independent of the real elastic restitution coefficient e of the beads. Note that for collapsed systems, the main source of dissipation does not originate in the viscosity associated with diffusion of momentum, but in collisional inelasticity, because the associated time is the shorter. The inelastic collapse phenomenon is also relevant to the explanation of the behavior of a single grain dropped onto a thick layer of particles. No bounce is observed, even in the case of a elastic restitution coefficient close to unity. The reason that is improper is to take total momentum as conserved in the center-of-mass frame of the two colliding grains involved in the nominal binary collision, because the whole substrate is involved in the momentum absorption process. An accounting of whole substrate (at rest) is required for a correct application of Newton's law. Since the total momentum before the collision of all involved bodies is practically zero, and since the apparent restitution coefficient is zero, we readily conclude that there is no bounce. This example serves to emphasize the necessity of considering *nonlocal* mechanisms for the momentum conservation in inelastic, dense media, in contrast to the standard derivation of the Navier-Stokes equation in hydrodynamic theory. In the following we attempt to model the rheological behavior of densely packed media on the basis of the previous considerations. We demonstrate that the linear velocity profile exhibited (in the steady regime) by collapsed granular media flowing down on incline results from the nonlocality of the momentum and the energy conservation.

Although a direct use of the Navier-Stokes equation is incorrect, it is nevertheless possible to gain insight into the rheological behavior of densely packed granular media by considering the energy balance. Let us consider an elementary shear transformation as sketched in Fig. 3. We assume that all losses originate in inelasticity. Consider then the collision between a grain having velocity $\mathbf{v}_i$ and belonging to the layer (i) and a grain having velocity $\mathbf{v}_{i+1}$ and belonging to the layer ($i+1$) beneath. In the case of binary collisions, the inelastic energy loss (per collision) is given by $E_{\text{sink}} = \frac{1}{4}m(1-e^2)(\mathbf{v}_i - \mathbf{v}_{i+1})^2$. This expression implies the conservation under collision of the total momentum and kinetic energy of the center of mass of the colliding particles. Thus this expression is

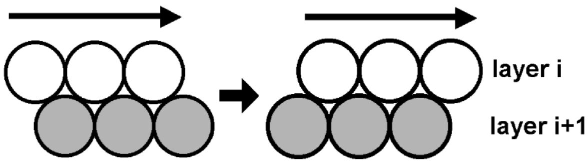

FIG. 3. Representation of an elementary shear transformation. The velocity of the upper layer (i) is $\mathbf{v}_i$, that of the layer ($i+1$) beneath is $\mathbf{v}_{i+1}$.

now incorrect, because the pair of grains considered is not isolated, and an appreciable fraction of the total momentum and kinetic energy is transmitted to surrounding particles. It is therefore more appropriate to identify the energy sink (partly mediated by nonlocal processes, such as collisional [19,20] or acoustic waves) with the total dissipation of the kinetic energy gained between consecutive collisions; that is, $E_{\rm sink} = \frac{1}{2}m(\mathbf{v}_i^2 - \mathbf{v}_{i+1}^2)$. Besides ensuring a steady regime, this expression accounts for the "sticking effect" experienced by a grain colliding with a collapsed cluster (i.e., for the full damping of the relative momentum), without entering into the details of the nonlocal dissipation processes. Let us adopt then the Eulerian description.

In the laboratory frame the decrease of the potential energy of one grain between two successive collisions reads $mgv(\nabla v)^{-1}$, since the frequency of collisions between adjacent layers is equal to ∇v. Accordingly, the kinetic energy dissipated per collision reads $E_{\rm sink} = \frac{1}{2}m(\mathbf{v}_i^2 - \mathbf{v}_{i+1}^2) \approx mv(d\nabla v)$. Equating the gain of kinetic energy between two collisions with the dissipation per collision yields for frictionless particles

$$\frac{\partial v_x}{\partial z} = \sqrt{\frac{g}{d}}. \tag{1}$$

The above derivation leads to the correct experimental result, namely, a constant shear rate with an order of magnitude $\sqrt{g/d}$, without predicting the coefficient of proportionality between ∇v and $\sqrt{g/d}$ which is found experimentally to range between 0.4 and 0.7. As will be shown further, an accounting for the sliding friction is required to recover the right experimental value. Note that the z-translational invariance found for the shear rate is also consistent with the z invariance found for other physical quantities, such as velocity fluctuations or density.

It is interesting to point out that within the postulated nonlocal dissipation scheme, the energy balance (per unit mass) reads

$$d(\nabla v)^2 v = gv \tag{2}$$

in the Eulerian description. This is to be compared with the relation $d^2(\nabla v)^3 \propto gv$ that would have been obtained from the usual local description assuming isolated binary collision, for homogeneous, isothermal media [the term as $(\nabla v)^3$ accounts both for viscous and for inelastic dissipation]. It is noteworthy that the left-hand term of Eq. (2) can be formally regarded as the rate of energy dissipated by the work of a volume force $\mathbf{F} = -\rho d(\nabla v)^2 \mathbf{v}/|\mathbf{v}|$. From this consideration, the following momentum equation (per unit mass) ensues:

$$\frac{Dv}{Dt} = g - d(\nabla v)^2 \tag{3}$$

(where D/Dt is the material derivative), to be compared with the standard Bagnold form $Dv/Dt = g - \alpha d^2[\partial(\nabla v)^2/\partial z]$.

Above we mapped the inelastic loss term into the work of an equivalent volume force $\mathbf{F}$. It can also be interpreted as the work of a shear stress $\sigma_{\rm coll}$ acting over the surfaces of an elementary volume. Reintroducing the slope angle θ, we get

$$\sigma_{\rm coll} = -\rho z d(\nabla v)^2 = -(d/g\cos\theta)p(\nabla v)^2, \tag{4}$$

where $p = \rho g z \cos\theta$ is the pressure. Although this standpoint is somewhat formal, the previous relation can be viewed as the constitutive relation relating stress to strain rate in a collapsed granular material flowing under gravity. Compared to the classical Bagnold result $\sigma \propto -\rho d^2(\nabla v)^2$, the new feature here is the linear dependence of the shear stress on the normal stress—*as with Coulomb solid friction.* Note that a closely related constitutive relation also depending both on the pressure and shear rate was heuristically proposed recently by Chevoir *et al.* to explain their experimental data [22].

So far we focused on the rheological behavior of frictionless particles. We aim now to address the role of intergranular sliding friction in the rheology. Torques originating in frictional contacts and acting on each particle can likely induce certain short range spatial correlations in grain rotations, but an investigation of this effect which occurs at microscopic scale is beyond the scope of this Letter. Instead, we wish here to concentrate on the issue of the macroscopic influence of the sliding friction on the shear rate. A noteworthy experimental result is that experiments conducted with highly frictional beads (with surface modified by chemical attack) also exhibit linear profiles of velocity in the dense limit [16,23]. The only noticeable effect of the increased intergranular friction is a decrease of the flow rate, which obviously results from the larger fraction taken by frictional loss in dissipating gravitational potential energy. The persistency of the linear velocity profile directly results from the rate independence of Coulomb friction. Frictional dissipation can indeed be shown to be *proportional* to the flow rate, *whatever* the velocity profile [16,23]. As a consequence Coulomb friction does not play any role in determining the velocity profile, which is actually determined by others constraints, such as the rate-dependent viscosity for dilute granular media, or the inelastic losses, in the case of collapsed systems.

Adding the Coulomb friction term, which reads (per unit volume) $\partial\sigma_{xz}/\partial z = k\rho g\cos\theta$ (where $k = \tan\theta_c$ is the coefficient of friction of the material) on the right-hand side of Eq. (4) readily yields

$$\frac{Dv}{Dt} = \frac{g}{\cos\theta_c}\sin(\theta - \theta_c) - d(\nabla v)^2, \qquad (5)$$

and we obtain hence

$$\nabla v = \left[\frac{\sin(\theta - \theta_c)}{\cos\theta_c}\right]^{1/2}\sqrt{\frac{g}{d}}, \qquad (6)$$

in the steady regime for the shear rate. Taking a typical value of 20° for θ_c leads to a value of the prefactor $[\sin(\theta - \theta_c)/\cos\theta_c]^{1/2}$ ranging between 0.4 and 0.7 in the range $30° < \theta < 50°$ investigated experimentally, which is consistent with data. We deduce for the flow rate the relation $Q = [g\sin(\theta - \theta_c)/d\cos\theta_c]^{1/2}(h^2/2)$ (where h is the depth of the flowing layer), which holds *provided that the condition of zero velocity* (nonsliding) *is satisfied at the bottom.* Experimentally, the dependence on shear rate indicated by Eq. (6) has been recognized recently by Orpe and Khakhar [24].

It is of interest to compare the energy $W_{\rm fric}$ dissipated by friction to that dissipated by inelastic collisions $E_{\rm inel}$. We find $(W_{\rm fric}/E_{\rm inel}) = g\tan\theta_c\cos\theta/d(\nabla v)^2$, and hence

$$\frac{W_{\rm fric}}{E_{\rm inel}} = \frac{\sin\theta_c\cos\theta}{\sin(\theta - \theta_c)}. \qquad (7)$$

Note the divergence as $(\theta - \theta_c)^{-1}$ in the limit $\theta \to \theta_c$. This implies that for most of particulate flows encountered in nature, for which θ is close to θ_c, the largest fraction of the dissipation originates in the Coulomb friction (although the shape of the velocity profile is governed by the inelasticity of the collisions). For $\theta_c = 20°$ and $\theta = 21°$, we find $(W_{\rm fric}/E_{\rm inel}) \approx 18$.

To conclude, we have revisited the usual assumptions surrounding the rheology of rapid granular flows, and we have shown that the Bagnold constitutive relation does not hold in the dense limit. Emphasizing the role of the inelastic collapse, we have proposed an alternative framework and have demonstrated first that the rheology of rapid, dense, frictionless flows is entirely governed by the inelastic dissipation, which is shown to be equivalent to the work of a volume force $|\mathbf{F}| = \rho d(\nabla v)^2$. On these bases, we have deduced the rheological behavior for densely packed free surface flow of frictional grains flowing down the inclined plane. We have determined that the rheology is independent of the elastic restitution coefficient, that the velocity profile is linear, and that the shear rate is proportional to $\sqrt{g/d}$. The factor of proportionality between ∇v and $\sqrt{g/d}$ depends only on the Coulomb friction and is equal to $[\sin(\theta - \theta_c)/\cos\theta_c]^{1/2}$. These results agree quantitatively with experimental measurements. Finally we have also showed that for dense particulate flows, the dissipation is mainly due to frictional sliding.

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