Vortex States of Rapidly Rotating Dilute Bose-Einstein Condensates

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We show that, in the Thomas-Fermi regime, the cores of vortices in rotating dilute Bose-Einstein condensates adjust in radius as the rotation velocity, Ω, grows, thus precluding a phase transition associated with core overlap at high vortex density. In both a harmonic trap and a rotating hard-walled bucket, the core size approaches a limiting fraction of the intervortex spacing. At large rotation speeds, a system confined in a bucket develops, within Thomas-Fermi, a hole along the rotation axis, and eventually makes a transition to a giant vortex state with all the vorticity contained in the hole.

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The energetically favored state of rotating superfluids such as ^{4}He [1] and the alkali gas Bose-Einstein condensates [2–4] is a triangular lattice of singly quantized vortices [5]. Here we pose the question of the structure of the lattice and ultimate fate of the superfluid at rotation speeds so large that a sizable fraction of the fluid is filled by the vortex cores. A type II superconductor with a vortex array simply becomes *normal* at a high magnetic field, H_{c2}, where the vortex cores begin to overlap [6]. While in a rotating Bose superfluid the cores of quantized vortex lines (if of fixed size) would completely fill the system at an upper critical rotation speed Ω_{c2} analogous to H_{c2}, a low temperature bulk bosonic system does not have a simple normal phase to which it can return, and thus the problem [7]. In He II the critical rotation rates at which the vortex cores approach each other are unobservably large, $\Omega_{c2} \sim 10^{12}$ rad/sec. In atomic dilute Bose-Einstein condensates, by contrast, inertial forces can be comparable to interatomic forces, and the approach to tightly packed vortex lattices is within reach of experiment; indeed current experiments in harmonically trapped gases achieve rotational velocities Ω that are a significant fraction of Ω_{c2} [2–4,8,9].

We consider a Bose gas at zero temperature in the Thomas-Fermi regime, with weak repulsive interactions described by an s-wave scattering length a_s, and integrate out (in the sense of the renormalization group) the short range structure on scales of the vortex separation. The structure of the cores and the destiny of the vortex lattice at high rotational speeds depend fundamentally on the geometry of the container confining the fluid. We study here the modification of the lattice induced by high rotational speeds in both harmonic traps and in cylindrical hard-walled (square-well) buckets [10]. In a harmonic trap, where considerable work has been done on understanding the states of the fluid [7,11], the radial trapping potential, $V_\perp = \frac{1}{2} m\omega_\perp^2 r^2$, where r is the perpendicular distance from the axis of rotation, dominates the centrifugal potential. As the rotational frequency Ω approaches $\omega_\perp$, the system becomes quasi-two-dimensional and should eventually enter a quantum Hall-like state [12–15]; we do not treat this limit here. We find that in a trap the cross-sectional area occupied by the vortex cores grows until they fill a limiting fraction, $\sim 1/2$, of the space of the system, and they never touch, even for rotational velocities arbitrarily close to $\omega_\perp$. This result goes beyond that of [7], in that we take into account (albeit approximately) the modification of the vortex core in the finite sized cell of the lattice, for a given particle number in the cell.

In a cylindrical bucket, on the other hand, where the centrifugal force tends to throw the fluid against the walls, we find that at a critical rotation speed, $\Omega_h = \sqrt{2}\hbar/m\xi_0 R$, the fluid begins to develop a cylindrical hole in the center which grows with increasing Ω; here R is the bucket radius, $\xi_0 = \hbar(2mg\bar{n})^{-1/2}$ is the coherence or healing length, $g = 4\pi\hbar^2 a_s/m$, and $\bar{n}$ is the uniform density in a nonrotating cloud. In the bucket, the ratio of the cross-sectional core area to the area of the unit cell of the vortex lattice is independent of Ω for $\Omega > \Omega_h$, and given by $9\xi_0/\sqrt{2}R$. A phase transition associated with core overlap at high vortex density is thus precluded for either trapping geometry. In addition, as we show, a system in a bucket eventually undergoes a phase transition in the limit of large Ω to a giant vortex phase with the vorticity concentrated along the cylinder axis.

The scale of rotation velocities in a dilute gas in a container is set by $\Omega_0 = 2g\bar{n}/\hbar = \hbar/m\xi_0^2 \sim \Omega_{c2}$. For $\xi_0 \simeq 0.2$ μm [2,3], $\Omega_0 \simeq 1.8 \times 10^4$ rad/sec in ^{87}Rb and $\simeq 6.9 \times 10^4$ rad/sec in ^{23}Na.

We describe the ground state of the system with a rotating vortex lattice by an order parameter, $\psi(\boldsymbol{r})$, determined by minimizing the energy, $E' = E - \Omega L_z$, in the rotating frame by means of a variational calculation; here L_z is the component of the angular momentum of the system along the rotation axis. With $\hbar = 1$,

$$E' = \int d^3 r \left[\frac{1}{2m} |(-i\nabla - m\boldsymbol{\Omega} \times \boldsymbol{r})\psi|^2 + \left(V(\boldsymbol{r}) - \frac{1}{2} m\Omega^2 r^2 \right) |\psi|^2 + \frac{1}{2} g |\psi|^4 \right], \quad (1)$$

where $V(\boldsymbol{r})$ is the trapping potential. For a single vortex line of unit winding number on the cylindrical axis, $\psi(\mathbf{r}) = f(r,z)e^{i\phi}$ in polar coordinates. To calculate E' we employ a Wigner-Seitz approximation [16], replacing the triangular shaped cells of the (2D) lattice by cylindrical cells of equal radius ℓ; the assumption of equal cell area is consistent with observed lattices in traps [3]. We assume straight vortex lines for simplicity, and that the lattice rotates at a uniform angular velocity Ω_v, related to the (2D) density of vortices $n_v = 1/\pi\ell^2$ by $\Omega_v = \pi n_v/m = 1/m\ell^2$; Ω_v is in general smaller than the stirring velocity Ω by terms of order $1/mR^2$.

In general, ψ in a cell is a unique function of the number of particles therein and can be constructed, e.g., by the method of matched asymptotic expansion [17]. It is sufficient here to use the simple approximation that within a cell, labeled by i,

$$f(x,z) = \begin{cases} (x/\xi)\sqrt{n_i(z)}, & 0 \le x \le \xi, \\ \sqrt{n_i(z)}, & \xi \le x \le \ell, \end{cases} \tag{2}$$

where x is the (polar) radial coordinate measured from the center $\mathbf{r}_i$ of the cell. To account for the density variation in the system we allow the mean density $n(\mathbf{r}_i, z)$ within a cell centered on $\mathbf{r}_i$ to depend on the radial position of the cell and height. Ansatz (2) is consistent with the exact boundary conditions, $f(x \to 0, z) \sim x$, and $df(x,z)/dx|_{x=\ell} = 0$. While the core radius in a cell should depend on the local density, for simplicity we take ξ to depend only on the mean density in the system, an approximation adequate to bring out the essential physics of rotation at high speeds in the Thomas-Fermi regime. As one approaches the mean-field quantum Hall regime [14], it becomes necessary to include possible dependence of f on the trapping frequency as well.

The mean density $n(\mathbf{r}_i, z)$ in the cell is

$$n(\mathbf{r}_i, z) = \frac{1}{\pi\ell^2}\int_0^\ell d^2r|\psi|^2 = n_i(z)(1-\zeta/2), \tag{3}$$

where $\zeta \equiv (\xi/\ell)^2 \le 1$ is the ratio of the core area to the transverse area of the unit cell of the lattice.

Writing the local velocity as $\mathbf{v} = \Omega_v r\hat{\boldsymbol{\phi}} + \delta\mathbf{v}$ to evaluate the kinetic energy in each cell, as in [18], we find

$$\begin{aligned} E &= \sum_i E_i \\ &\to \int d^3rn(\mathbf{r})\Big(\frac{1}{2}m\Omega_v^2r^2 + (a+1)\Omega_v + V(\mathbf{r}) \\ &\qquad + \frac{1}{2}gn(\mathbf{r})b\Big), \end{aligned} \tag{4}$$

where we replace the sum over cells by a continuous integration, with $\mathbf{r}_i \to \mathbf{r}$. In Eq. (4) the kinetic energy per cell is $\pi a(\zeta)n(\mathbf{r})/m$ and the interaction energy per cell is $gn^2(\mathbf{r})b(\zeta)\pi\ell^2/2$, where

$$\begin{aligned} a &= \frac{1}{n(\mathbf{r})}\int_0^\ell dxx\Big[\Big(\frac{f}{x}\Big)^2 + \Big(\frac{\partial f}{\partial x}\Big)^2\Big] = \frac{1-\ln\sqrt{\zeta}}{1-\zeta/2}, \\ b &= \frac{2}{\ell^2n^2(\mathbf{r})}\int_0^\ell dxxf^4(x,z) = \frac{1-2\zeta/3}{(1-\zeta/2)^2}; \end{aligned} \tag{5}$$

$a(\zeta)$ has a minimum at $\zeta \simeq 0.56$, while $b(\zeta)$ increases slowly with ζ. The first term in (4) is the energy of solid body rotation of the lattice, while the term $\sim n(\mathbf{r})\Omega_v$ arises from $\Omega_v r\hat{\boldsymbol{\phi}}\cdot\delta\mathbf{v}$. The result (4) is effectively the Thomas-Fermi approximation for the coarse grained structure, described by $n(\mathbf{r})$, with the functions a and b accounting for the fine grained vortex structure.

The total angular momentum in the Wigner-Seitz approximation is [18]

$$L_z = \int d^3rn(\mathbf{r})(1 + \Omega_v mr^2), \tag{6}$$

the sum of the angular momentum per vortex cell and the solid body rotation contribution; L_z is independent of a and b. *In toto,*

$$\begin{aligned} E' = \int d^3rn(\mathbf{r})\Big[\Big(\frac{\Omega_v^2}{2} - \Omega_v\Omega\Big)mr^2 \\ + (a+1)\Omega_v - \Omega + V(\mathbf{r}) + \frac{1}{2}gn(\mathbf{r})b\Big]. \end{aligned} \tag{7}$$

Minimization of Eq. (7) with respect to Ω_v implies that $\Omega_v = \Omega - (a+1)N/I$, where $I = \int d^3rmr^2n(\mathbf{r})$ is the moment of inertia. For large systems, $mR^2\Omega \gg 1$, we neglect the small difference of Ω_v and Ω; then

$$E' = \int d^3rn(\mathbf{r})\Big[-\frac{\Omega^2}{2}mr^2 + V(\mathbf{r}) + a\Omega + \frac{1}{2}gn(\mathbf{r})b\Big]. \tag{8}$$

Minimization of Eq. (8) with respect to $n(\mathbf{r})$ at fixed total particle number yields the Thomas-Fermi result,

$$n(\mathbf{r}) = [\tilde{\mu} + m\Omega^2r^2/2 - V(\mathbf{r})]/gb, \tag{9}$$

where μ is the chemical potential and $\tilde{\mu} = \mu - a\Omega$. In a trap with transverse frequency $\omega_\perp$ and longitudinal frequency ω_z, the density has the usual form [19]

$$n_{\rm trap}(\mathbf{r}) = (\tilde{\mu}/gb)[1 - (r/R_t)^2 - (z/Z_t)^2], \tag{10}$$

with $R_t^2 = (2\tilde{\mu}/m)/(\omega_\perp^2 - \Omega^2)$ and $Z_t^2 = (2\tilde{\mu}/m)/\omega_z^2$. Then $N = 8\pi Z_tR_t^2\tilde{\mu}/15gb$ and

$$\frac{\tilde{\mu}}{\omega_\perp} = \frac{1}{2}\Big[15Nba_s\sqrt{m\omega_\perp}\frac{\omega_z}{\omega_\perp}\Big(1 - \frac{\Omega^2}{\omega_\perp^2}\Big)\Big]^{2/5}. \tag{11}$$

By contrast, the density in a cylinder grows quadratically with r, until at a critical velocity Ω_h where $\tilde{\mu} = 0$, the system develops a cylindrical hole in the center; in a trap the quadratic trapping potential dominates, preventing formation of a hole. Integrating Eq. (9) over the volume of the cylinder, we find that the hole begins to develop at $\Omega_h = \sqrt{2b}\Omega_0\xi_0/R = (2b\Omega_0/mR^2)^{1/2}$. Then

$$n_{\rm cyl}(r) = \begin{cases} \bar{n} + m\Omega^2(r^2 - R^2/2)/2gb, & \Omega < \Omega_h, \\ m\Omega^2(r^2 - R_h^2)/2gb, & \Omega \geq \Omega_h; \end{cases} \quad (12)$$

where the radius of the hole is $R_h = R\sqrt{1 - \Omega_h/\Omega}$. The phase of the order parameter winds by $2\pi m\Omega R_h^2$ at the inner edge of the hole, corresponding to an array of phantom vortices of density n_v inside the hole.

The energy per particle in the rotating frame is

$$\frac{E'_{\rm trap}}{N} = \frac{5\tilde{\mu}}{7} + a\Omega, \quad (13)$$

$$\frac{E'_{\rm cyl}}{N} = -\frac{mR^2\Omega^2}{4} + a\Omega + \frac{\Omega_0 b}{4} - \frac{b\Omega_0}{12}\left(\frac{\Omega}{\Omega_h}\right)^4, \quad \Omega < \Omega_h$$
$$\frac{E'_{\rm cyl}}{N} = -\frac{mR^2\Omega^2}{2} + a\Omega + \frac{m\Omega R^2\Omega_h}{3}, \quad \Omega \geq \Omega_h. \quad (14)$$

The core size is determined by minimizing E' with respect to ζ at fixed N and Ω; differentiating Eq. (8) directly we have

$$\Omega\frac{\partial a}{\partial \zeta} + \frac{g\langle n\rangle}{2}\frac{\partial b}{\partial \zeta} = 0, \quad (15)$$

where $\langle n\rangle = \int n^2(r)/\int n(r)$ is a mean of the density in the system. [Note that were we, more correctly, to solve for the vortex structure within each cell, then the integration in $\langle n\rangle$ in Eq. (15) would be only over the given cell.] In a bucket the relevant regime is small ζ, where $b(\zeta) \simeq 1 + \zeta/3$, and $\partial a/\partial\zeta \simeq -1/2\zeta$, the minimum is thus at $\zeta_{\rm cyl} = 3\Omega/g\langle n\rangle$. For $\Omega < \Omega_h$, $\langle n\rangle = \bar{n}[1 + \frac{1}{3}(\Omega/\Omega_h)^4]$, and

$$\zeta_{\rm cyl} \simeq \frac{6\Omega/\Omega_0}{1 + \frac{1}{3}(\Omega/\Omega_h)^4}, \quad \Omega < \Omega_h. \quad (16)$$

For $\Omega \geq \Omega_h$, $\langle n\rangle = \frac{4}{3}\bar{n}\Omega/\Omega_h$, and

$$\zeta_{\rm cyl} = \frac{9}{2}\frac{\Omega_h}{\Omega_0}, \quad \Omega \geq \Omega_h. \quad (17)$$

The total area occupied by the vortex cores thus scales as the ratio of zero temperature coherence length and system size and is independent of Ω; see Fig. 1(a).

In a trap, $\langle n\rangle = 4\tilde{\mu}/7gb$; for small Ω, $\zeta \ll 1$ and $\partial a/\partial\zeta \simeq -1/2\zeta$, so that ξ grows with increasing Ω as $(\omega_\perp^2 - \Omega^2)^{-1/5}$, as in [7]. However, as $\Omega \to \omega_\perp$, $\tilde{\mu}$ and therefore $\langle n\rangle \to 0$, and the solution of (15) requires $\partial a/\partial\zeta \to 0$, which occurs at $\zeta \simeq 1/2$; this is the limiting value of ζ. Quite generally, the fact that $\langle n\rangle \to 0$ as $\Omega \to \omega_\perp$ implies that the solution must minimize the kinetic energy in Eq. (1). The cores can never overlap, and the system can, within mean-field theory, maintain a lattice of quantized vortices for all $\Omega < \omega_\perp$. This behavior is shown in Fig. 1(b); in strongly elongated traps, as in a cylinder, the system reaches a *self-similar* regime in which the core size scales with the intervortex spacing.

The rotation speeds needed to observe the increase of core size with Ω are in principle accessible within current harmonic trap experiments, prior to the onset of a quantum Hall state. In [3], $N \sim 5\times 10^7$, $\omega_\perp/\omega_z = 4.2$, $\Omega/\omega_\perp \lesssim 0.7$, and $\omega_\perp/\Omega_0 = 0.008$; an increase in aspect ratio by a factor of 24, corresponding to the solid curve in Fig. 1(b), should be sufficient to observe the strong increase of the core size for $\Omega \lesssim \omega_\perp$.

At sufficiently large Ω, a dilute gas in a cylindrical trap makes a transition to a giant vortex state, with vorticity concentrated in the center rather than spread throughout the system. Such behavior is seen in numerical simulations in traps that rise faster than harmonic [20,21] (and also in the presence of vortex pinning [22]). Dynamical formation of metastable giant vortices in harmonic traps with 7 to 60 units of vorticity is reported in Ref. [23]. At the level of Thomas-Fermi, which is adequate to describe the giant vortex regime in large systems ($mR^2\Omega \gg 1$) a giant vortex, described by order parameter $\psi(\mathbf{r}) = \sqrt{n_{\rm G}(r)}e^{i\nu\phi}$, has an energy in the rotating frame,

$$E'_{\rm G} = \int d^3r\, n_{\rm G}(\mathbf{r})\left(\frac{\nu^2}{2mr^2} + \frac{g}{2}n_{\rm G}(\mathbf{r}) - \nu\Omega\right). \quad (18)$$

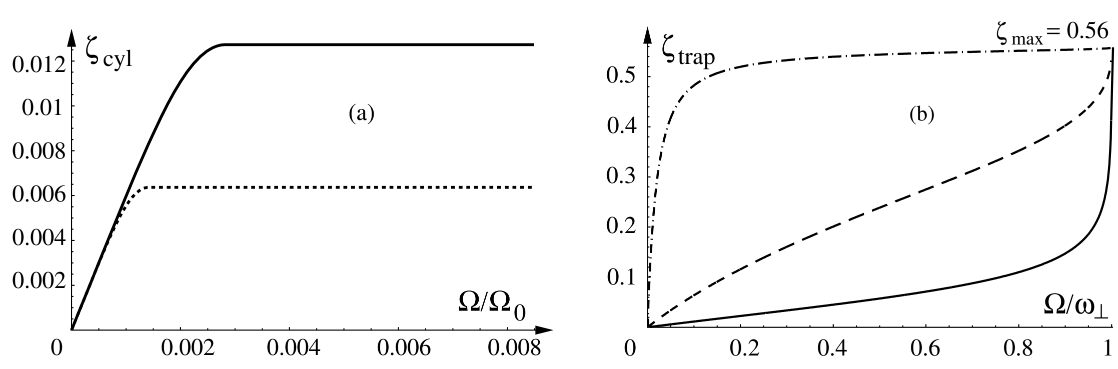

FIG. 1. Vortex core area as a fraction of the Wigner-Seitz cell area: (a) ζ vs Ω/Ω_0 in a hard-walled bucket with $R/\xi_0 = 500$ (solid line) and $R/\xi_0 = 1000$ (dashed line); (b) ζ vs $\Omega/\omega_\perp$ in an anisotropic cylindrically symmetric harmonic trap with parameters $N = 5\times 10^7$, $a_s/d_\perp = 0.01$, and $\omega_\perp/\omega_z = 10^2$ (solid line), 10^4 (dashed line), and 10^8 (dash-dotted line).

Minimizing with respect to $n(\mathbf{r})$ we find the analog of Eq. (9), $n_G(\mathbf{r}) = (\tilde{\mu} - \nu^2/2mr^2)/g$, where $\tilde{\mu} = \mu + \nu\Omega$. In Thomas-Fermi, the giant vortex has a hole in the center of radius $R_G = \alpha R$, where $\alpha = \nu/R\sqrt{2m\tilde{\mu}}$. In the limit of large Ω ($\gg \Omega_h$), where $\alpha \to 1$, we expand in powers of $(\Omega_0/mR^2\Omega^2)^{1/2}$ and find

$$\frac{E'_G}{N} = -\frac{1}{2}mR^2\Omega^2 + \frac{\sqrt{2mR^2\Omega_0}}{3}\Omega + \frac{\Omega_0}{18} + \cdots. \quad (19)$$

Comparing with the lattice energy, Eq. (14), and using (17), we find the transition to the giant vortex state at

$$\Omega_G = \frac{\Omega_0}{9[1 + 2a(9\xi/\sqrt{2}R)]} \simeq \frac{\Omega_0}{9[1 + \ln(R/\xi_0)]}. \quad (20)$$

In this Letter we have studied only the zero temperature structure of rotating Bose gases in the long wavelength Thomas-Fermi regime. The full structure at finite temperatures reflected in the phase diagram in the Ω-T plane (studied for a harmonically trapped gas in [24]) may be rich. For example, at sufficiently large rotation velocities and low temperatures such that thermally excited motion perpendicular to the boundary becomes frozen out, the fluid forced against the boundary of the bucket becomes effectively two dimensional. Under these circumstances the fluid can possibly undergo a transition to a two-dimensional Kosterlitz-Thouless phase.

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