Decoherence-Free Generation of Many-Particle Entanglement by Adiabatic Ground-State Transitions

R. G. Unanyan[1,2] and M. Fleischhauer[1]

[1]*Fachbereich Physik, Universität Kaiserslautern, 67653 Kaiserslautern, Germany*

[2]*Institute for Physical Research, Armenian National Academy of Sciences, 378410 Ashtarak, Armenia*

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We discuss a decoherence insensitive method to create many-particle entanglement in a spin system with controllable collective interactions and propose an implementation in an ion trap. An adiabatic change of parameters allows a transfer from separable to a large variety of entangled eigenstates. We show that the Hamiltonian can have a supersymmetry permitting an explicit construction of the ground state at all times. Of particular interest is a transition in a *nondegenerate* ground state with a finite energy gap since here the influence of collective as well as individual decoherence mechanisms is substantially reduced. A lower bound for the energy gap is given.

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Entanglement is one of the most characteristic features of quantum systems and lies at the heart of quantum information processing and computing [1]. While in few-particle systems entanglement is by now reasonably well understood [2] and practical schemes for its creation and manipulation are developed [3,4], many-particle entanglement is still an open field of research with a large unexplored potential for applications. There exist, for example, proposals to implement quantum computation in an initially entangled many-particle system by performing only measurements and single qubit operations, both of which are relatively easy to implement [5]. A necessary prerequisite for this is, however, a specific many-particle entanglement. Since entangled states become increasingly susceptible to environmental interactions if the number of particles increases, an important practical challenge is the design of robust and most importantly decoherence-resistant mechanisms for its generation. We here propose and analyze such a mechanism which is based on adiabatic transitions in a spin system with controllable collective interactions. Robustness against decoherence and short process times are achieved by choosing ground-state transitions with a large energy gap very similar to the ideas used in adiabatic quantum computation [6].

Let us consider a collection of N interacting spin $1/2$ systems described by a generalization of the Lipkin-Meshkov-Glick (LMG) Hamiltonian [7]

$$H = \xi[\lambda\chi_1\chi_2\hat{J}_z + \chi_1^2\hat{J}_x^2 + \chi_2^2\hat{J}_y^2 + 2\mu\chi_2^2\hat{J}_y], \qquad (1)$$

where the $\hat{J}_i$ $(i \in \{x, y, z\})$ are the total-spin operators of the ensemble. We show that this Hamiltonian can be implemented for $\mu = 0$ with controllable parameters by a generalization of the ion-trap scheme suggested by Sørensen and Mølmer [8], where cold ions interact via a common trap oscillation and are driven by bichromatic laser fields. In contrast to the effective Hamiltonian of [8] the generalized LMG interaction (1) cannot be solved exactly. It does provide, however, the possibility for adiabatic and therefore robust transitions between separable and entangled many-particle states. The most important feature of (1) is that such transitions are possible while staying in a *nondegenerate* ground state with a finite energy gap and that a large class of entangled target states are accessible by varying μ. We show that in the special case of $\lambda = 1$, the LMG Hamiltonian (1) has a supersymmetry and the ground state can be explicitly constructed for all times of the transfer process. The gap can be made rather large and hence the process can be fast despite its adiabatic nature. For the same reason the influence of decoherence is strongly reduced: *Collective* decoherence due to noise in the external control parameters is eliminated by the adiabatic nature of the process, and the influence of *independent* individual reservoir couplings is suppressed due to the presence of a finite energy gap.

Let us first discuss the case of a negative coupling parameter $\xi < 0$. If $\mu = 0$ and $\lambda \geq N$, the system described by Eq. (1) undergoes a quantum-phase transition [9] when the interaction parameters χ_1 and χ_2 are changed [case (i)]. This transition can be described analytically in the semiclassical limit with $J = N/2$ and $N \gg 1$. If $\chi_1 = \chi_2$, $\hat{J}_z$ is a conserved quantity and (1) has a trivial anharmonic spectrum: $H \rightarrow \xi\chi_2^2[\lambda\hat{J}_z - \hat{J}_z^2 + \hat{\mathbf{J}}^2]$. The ground state of the system has maximum total angular momentum $J = N/2$ and a z projection $m_z = \pm N/2$ depending on the sign of λ. Both of these states denoted by $|\uparrow\uparrow\ldots\rangle$ and $|\downarrow\downarrow\ldots\rangle$ are separable many-particle states. On the other hand, in the limit $\chi_1 = 0$ the terms in (1) containing $\hat{J}_x$ and $\hat{J}_z$ vanish and the Hamiltonian approaches $H \rightarrow \xi\chi_2^2\hat{J}_y^2$. It has again a trivial spectrum, however, with two *degenerate* ground states $|m_y = \pm N/2\rangle$. Both of these, taken individually, are separable. Symmetric or antisymmetric superpositions of them form, however, the N-spin analogs of the Greenberger-Horne-Zeilinger (GHZ) states. Using the Schwinger-representation of angular momenta [10] one finds that the state $|m_z = N/2\rangle$ is adiabatically connected

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to only one particular superposition due to the symmetry of the Lipkin interaction. Thus

$$|\uparrow\uparrow\ldots\rangle \xrightarrow{\text{(i)}} \frac{1}{\sqrt{2}}[|++\ldots\rangle + e^{i\pi J}|--\ldots\rangle], \tag{2}$$

which corresponds to the generation of the N-particle analog of the GHZ state in the σ_y basis. $|\pm\rangle$ denote single-particle eigenstates of σ_y with $m_y = \pm 1/2$, respectively. Because of the phase factor $e^{i\pi J}$ the entangled state depends sensitively on the total number of atoms even in the limit $N \to \infty$ [11].

For a positive coupling parameter, $\xi > 0$, three cases need to be distinguished: $\mu = 0$ and the total number of spins is odd [case (ii)]; $\mu = 0$ and the total number of spins is even [case (iii)]; and $\mu \neq 0$ [case (iv)]. In all cases the ratio χ_1/χ_2 is again rotated from unity to zero.

Since $\xi > 0$ the initial ground state in cases (ii) and (iii) is $|m_z = \pm N/2\rangle$ depending on the sign of $\lambda \neq 0$ and is separable. The final Hamiltonian is again $H = \xi\chi_2^2\hat{J}_y^2$, whose ground state is now, however, the eigenstate of $\hat{J}_y$ with smallest value of $|m_y|$. Thus for an odd number of particles [case (ii)] there are two *degenerate* ground states with $m_y = \pm 1/2$, both of them being maximally entangled. Making use of the symmetry of the interaction and using the Schwinger representation we find that the adiabatic transition leads to the mapping

$$|\downarrow\downarrow\ldots\rangle \xrightarrow{\text{(ii)}} \frac{1}{\sqrt{2}}[|(+)^n(-)^{n-1}\rangle_s + i|(+)^{n-1}(-)^n\rangle_s], \tag{3}$$

where $n = \lceil N/2 \rceil$ and the subscript "s" denotes symmetrization.

In case (iii) the final ground state ($\chi_1 \to 0$) is *nondegenerate,* has spin projection $m_y = 0$, and is maximally entangled.

$$|\downarrow\downarrow\ldots\rangle \xrightarrow{\text{(iii)}} |m_y = 0\rangle = |(+)^{N/2}(-)^{N/2}\rangle_s. \tag{4}$$

In this case there is no merging of eigenstates and consequently no phase transition. The absence of degeneracy during the entire adiabatic transfer makes case (iii) particularly interesting because here decoherence is strongly suppressed by the presence of a finite energy gap. The target state is, however, fixed to the special case $|m_y = 0\rangle$.

The variety of accessible target states in a nondegenerate adiabatic ground-state transition can be substantially increased by adding a linear interaction proportional to $\hat{J}_y$ to the LMG Hamiltonian, i.e., by allowing for a nonvanishing value of μ in Eq. (1) [case (iv)]: If we assume for simplicity that $\mu = m$, with m being an integer or half-integer with $m \in \{-J, -J+1, \ldots, J\}$, the final ($\chi_1 \to 0$) Hamiltonian approaches $H \to \xi\chi_2^2[\hat{J}_y^2 + 2m\hat{J}_y]$. Its ground state is $|m_y = m\rangle$ and thus can be adjusted to have *any* eigenvalue of $\hat{J}_y$, which is maximally entangled unless $m = \pm J$.

$$|\downarrow\downarrow\ldots\rangle \xrightarrow{\text{(iv)}} |m_y = m\rangle. \tag{5}$$

In all four cases considered an adiabatic change of χ_1/χ_2 from unity to zero leads to the generation of a well defined entangled state. In cases (iii) and (iv) this is, moreover, possible with a finite energy gap. The adiabatic process is neither of the Landau-Zener nor of the type of stimulated Raman adiabatic passage [12]. We have illustrated the different scenarios for even N, $\mu = 0$, and a positive linear term in (1) in Fig. 1.

To find the ground state of (1) for arbitrary values of χ_1/χ_2 as, e.g., during the transfer process is a difficult task. We will show now that (1) possesses a supersymmetry (SUSY) for $\lambda = 1$, allowing for an explicit construction of the ground state for cases (iii) and (iv). (The existence of an extra symmetry has been suspected [13] but has not been understood so far.) For $\lambda = 1$ the Hamiltonian can be factorized as

$$\frac{H}{\xi} = (\chi_1\hat{J}_x + i\chi_2\hat{J}_y - i\chi_2\mu)(\chi_1\hat{J}_x - i\chi_2\hat{J}_y + i\chi_2\mu) - \chi_2^2\mu^2. \tag{6}$$

Because of the SUSY the spectrum consists of twofold degenerate and a nondegenerate state [14]. For $\mu = m \in \{-J, \ldots, J\}$ the ground state is nondegenerate and obeys $(\chi_1\hat{J}_x - i\chi_2\hat{J}_y + i\chi_2 m)|\psi_0\rangle = 0$. One finds

$$|\psi_0\rangle = \mathcal{N}\exp(\gamma\hat{J}_z)|m_y = m\rangle, \tag{7}$$

where $\mathcal{N}$ is a normalization constant and $\tanh(\gamma) = \chi_1/\chi_2$ for $\chi_1 \leq \chi_2$. $|\psi_0\rangle$ is an entangled state of the N spins for any value of $\gamma \neq 0$. That is, by varying γ and choosing m, we have access to a rich variety of entangled many-particle states while staying in the lowest energy state.

We proceed by discussing the conditions for adiabaticity and the sensitivity of the process to decoherence. In this context particular attention has to be given to the

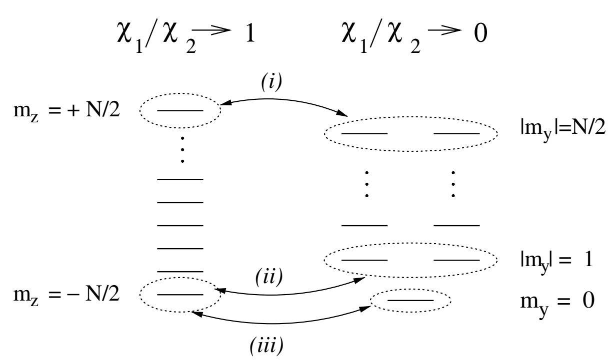

FIG. 1. Energy spectrum of Hamiltonian (1) for $\xi > 0$, even number of particles N, and maximum J in the limits for $\chi_1/\chi_2 \to 1$ and $\lambda > 0$ (left) and $\chi_1/\chi_2 \to 0$ (right). If the sign of ξ is changed, the picture has to be flipped upside down, and if the number of spins is odd [case (ii)], the state $m_y = 0$ does not exist and the state $|m_y| = 1$ needs to be replaced by $|m_y| = 1/2$. Adiabatic change of χ_1/χ_2 allows for robust transfer between separable (left) and entangled eigenstates (right) following the possible scenarios (i)–(iii).

phase transitions in cases (i) and (ii) since they are associated with a merging of pairs of energies. Because of the symmetry of (1) only one superposition of the associated states is, however, coupled to the nondegenerate initial ground state. Consequently, for the transition to be adiabatic it is sufficient that the characteristic time of the transfer T is much larger than the typical inverse frequency difference $\hbar/\Delta E$ to the next excited states. For the same reason *collective* decoherence processes caused, e.g., by fluctuations in the external parameters ξ, λ, χ_1, and χ_2 are suppressed in the present system by an exponential Boltzmann factor $\exp(-\Delta E/k_B T)$, where $k_B T$ is the thermal energy of the heat bath. Decoherence processes caused by *independent* heat bath couplings of the individual spins do not have the symmetry of (1), however, and hence do couple the degenerate states in cases (i) and (ii). Only in cases (iii) and (iv), i.e., without a quantum-phase transition, there is always a finite energy gap to all other states with the same total angular momentum J. Thus the entanglement generation in cases (iii) and (iv) will be robust against collective *and* individual decoherence processes, provided the energy gap is sufficiently large.

It is not possible to give an analytic expression for the energy gap ΔE in the general case. Numerical investigations for up to 50 particles indicate that for a transfer efficiency close to 100% it is sufficient that

$$\chi_1^{\max}\sqrt{T} \sim \chi_2^{\max}\sqrt{T} \gg 1 \tag{8}$$

with $\lambda \geq N$ for case (i) and $\lambda \geq 1$ for cases (ii)–(iv). T is the characteristic transfer time. An estimate for ΔE in cases (ii) and (iii), i.e., for $\mu = 0$, can be obtained as follows: Using a variational method with trial functions $|\Psi^{(N)}\rangle$, one finds an energy estimate $\langle H\rangle_N$ for the ground state with $J = N/2$, $E_0^{(N)} \leq \langle H\rangle_N$. Second, one can apply Temple's formula [15] to obtain a lower bound $E_0^{(N)} \geq \langle H\rangle_N - [\langle\Delta H^2\rangle_N/(E_1^{(N)} - \langle H\rangle_N)]$, where $\langle\Delta H^2\rangle_N = \langle(H - \langle H\rangle)^2\rangle_N$ is the energy fluctuation in the trial state and $E_1^{(N)}$ is the energy of the first excited state. One can show that Temple's formula gives the best lower bound when using only $\langle\Delta H^2\rangle_N$, $\langle H\rangle_N$, and $E_1^{(N)}$ as parameters. Furthermore, we make use of the inequality $E_1^{(N)} \geq E_0^{(N-2)}$ between the energy of the first excited state for N particles and the ground state for $N-2$ particles, in both cases with maximum J, which can easily be proven. Applying this inequality iteratively leads to

$$\frac{\Delta E^{(N)}}{\langle H\rangle_{N-2} - \langle H\rangle_N} \geq \frac{1}{2}(1 + \sqrt{1-4A}), \tag{9}$$

where

$$A > \frac{\langle\Delta H^2\rangle_N}{(\langle H\rangle_{N-2} - \langle H\rangle_N)(\langle H\rangle_{N-4} - \langle H\rangle_{N-2})}. \tag{10}$$

By choosing the trial function of the ground state close to the exact one it is possible to achieve $4A \ll 1$, i.e., $\Delta E^{(N)} \sim \langle H\rangle_{N-2} - \langle H\rangle_N$. For the simple trial function $|\Psi^{(N)}\rangle = \alpha_1|m_y = 0\rangle + \alpha_2|m_z = -N/2\rangle$ one finds $\langle H\rangle_{N-2} - \langle H\rangle_N = \beta\lambda|\xi|\chi_1\chi_2|_{\max}$, where β is a numerical factor of order unity, which varies only very slowly with N. Thus a reasonable estimate for the energy gap in case (iii) is given by $\beta\lambda|\xi|\chi_2^2$. This is also confirmed by our numerical calculations for particle numbers up to $N = 50$. If $\beta\lambda|\xi|\chi_2^2$ is sufficiently larger than the thermal energy of the environment, the probability of decoherence processes is strongly suppressed.

Let us now discuss a possible implementation of the Lipkin Hamiltonian (1) with $\mu = 0$ in an ion-trap system. Consider a linear trap with a string of ions with two relevant internal levels $|g\rangle$ and $|e\rangle$. The ions are assumed to be cooled such that only the in-phase collective oscillation of all ions is excited. The corresponding oscillation frequency is denoted by ν. The two internal levels of the ions are coupled by two laser fields with frequencies ω_1 and ω_2, and slowly varying Rabi frequencies Ω_1 and Ω_2. Assuming that both fields couple all ions in the same way, we can describe the system by the Hamiltonian $H = H_0 + H_{\text{int}}$, where $H_0 = \hbar\nu\hat{c}^\dagger\hat{c} + \hbar\omega_{eg}\hat{J}_z$, $\hat{c}$ and $\hat{c}^\dagger$ being the annihilation and creation operators of the trap oscillation. $\hbar\omega_{eg}$ is the energy separation between the two internal states and $\hat{J}_z = \frac{1}{2}\sum_{i=1}^N(\sigma_{ee}^i - \sigma_{gg}^i)$, with $\sigma_{\mu\mu}^i = |\mu\rangle_{ii}\langle\mu|$ being the projector to the internal state $|\mu\rangle$ of the ith ion. The interaction Hamiltonian H_{int} is given in rotating wave approximation by

$$H_{\text{int}} = \hat{J}_+ e^{i\eta(\hat{c}+\hat{c}^\dagger)}[\Omega_1 e^{-i\omega_1 t} + \Omega_2 e^{-i\omega_2 t}] + \text{H.c.}, \tag{11}$$

where η is the Lamb-Dicke parameter. We assume that the laser frequencies have equal and opposite detuning δ from resonance $\omega_{1,2} = \omega_{eg} \pm \delta$. δ is assumed to be large compared to the linewidth of the resonance but sufficiently different from the frequency of the trap oscillation, i.e., $|\delta|, |\nu \pm \delta| \gg \gamma$. As a consequence, the dominant processes are two-photon transitions leading to a simultaneous excitation of pairs of ions as indicated in Fig. 2. For $\Omega_1 = \Omega_2$ this scheme has first been considered by Sørensen and Mølmer [8,16] in the context

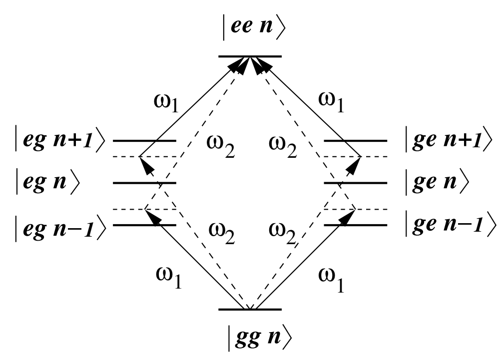

FIG. 2. Excitation scheme of pair of ions with ground state $|g\rangle$ and excited state $|e\rangle$ by bichromatic laser fields of equal and opposite detuning δ. n denotes the quantum number of trap oscillation.

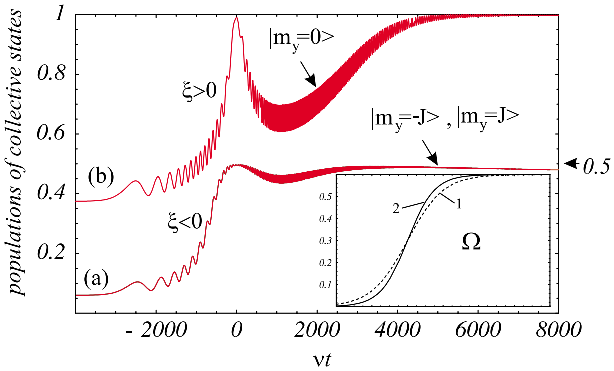

FIG. 3 (color online). Numerical simulation of the adiabatic transfer for a system of four ions. Shown are populations in states $|m_y = 0\rangle$ and $|m_y = \pm J\rangle$ for $\Omega_{1,2}(t) = (\alpha/2) \times [\tanh(t/T_{1,2}) + 1]$ as shown in the inset and for $\alpha/\nu = 0.6$, $\nu T_1 = 2000$, $\nu T_2 = 1500$, and $\delta/\nu = 0.9$ (a), case (i); and $\delta/\nu = 1.1$ (b), case (iii).

of quantum computation and dynamical entanglement generation.

We now assume that the ion trap is in the Lamb-Dicke limit, i.e., that the ions are cooled sufficiently enough, such that for all relevant excitation numbers n of the trap oscillation $(n + 1)\eta^2 \ll 1$ holds. In this limit one can expand the exponent in (11) to first order in η. For large values of $|\delta|$ it is convenient to consider this interaction in terms of a coarse-grained Hamiltonian which neglects the effects of rapidly oscillating terms. Using the time-averaging method of Ref. [17] one arrives at an effective Lipkin Hamiltonian (1) with the identifications $\xi = (2\nu\eta^2)/(\delta^2 - \nu^2)$, $\lambda = 2/\xi\delta$, and $\chi_{1,2} = \Omega_1 \mp \Omega_2$. $\Omega_1 = \Omega_2$ corresponds to the case $\chi_1 = \mu = 0$ in (1) which is exactly solvable and has been discussed in Refs. [8,16]. It has also been shown in [8,16] that the effective Hamiltonian describes correctly the dynamics of the ions in a coarse-grained time picture. Furthermore, the coupling scheme has successfully been implemented in experiments to generate entanglement between four ions [4].

In Fig. 3 we have shown as an example the effective dynamics of a system of four ions driven by fields Ω_1 and Ω_2 for the cases (i) and (iii) following Eq. (1) with $\mu = 0$. One recognizes that a nearly perfect transfer is possible.

In summary, we have shown that it is possible to generate specific entangled many-particle states in an ensemble of spins interacting through a collective coupling of the Lipkin-Meshkov-Glick type by adiabatic ground-state transitions. Two scenarios, (i) and (ii), involve a twofold degenerate ground state at some stages while two others, (iii) and (iv), always have a nondegenerate ground state. In all cases there is a finite energy gap to other excited states. This gap can be rather large, and thus fast processes are possible despite the required adiabaticity. In the asymptotic limits of the adiabatic transfer, the spectrum and eigenstates of the LMG Hamiltonian can be exactly calculated. Furthermore, for $\lambda = 1$ there is a supersymmetry allowing for an explicit construction of the ground state for all times. Because of adiabaticity, all transitions are robust against parameter variations. Furthermore, due to the symmetry of the coupling and the finite energy gap from the (degenerate or nondegenerate) ground state to excited states, *collective* decoherence processes are suppressed. In addition, in the case of a nondegenerate ground state [cases (iii) and (iv)] also *individual* decoherence processes are suppressed. The collective spin Hamiltonian can be implemented in a cold ensemble of ions in a linear trap driven by nearly resonant bichromatic fields.

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