Quantum Computation with Unknown Parameters

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We show how it is possible to realize quantum computations on a system in which most of the parameters are practically unknown. We illustrate our results with a novel implementation of a quantum computer by means of bosonic atoms in an optical lattice. In particular, we show how a universal set of gates can be carried out even if the number of atoms per site is uncertain.

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Scalable quantum computation requires the implementation of quantum gates with a very high fidelity. This implies that the parameters describing the physical system on which the gates are performed have to be controlled with a very high precision, something which is very hard to achieve in practice. In fact, in several systems only very few parameters can be very well controlled, whereas others possess larger uncertainties. These uncertainties may prevent current experiments from reaching the threshold of fault tolerant quantum computation (i.e., gate fidelities of the order of 0.9999 [1]), that is, the possibility of building a scalable quantum computer. For example, in quantum computers based on trapped ions or neutral atoms [2], the relative phase of the lasers driving a Raman transition can be controlled very precisely, whereas the corresponding Rabi frequency Ω has a larger uncertainty $\Delta\Omega$. If we denote by T the time required to execute a gate (of the order of Ω^{-1}), then a high gate fidelity requires $T\Delta\Omega \ll 1$ (equivalently, $\Delta\Omega/\Omega \ll 1$), which may be very hard to achieve, at least to reach the above mentioned threshold.

In this Letter we show how to achieve a very high gate fidelity even when most of the parameters describing the system cannot be adjusted to precise values. Our method is based on the technique of adiabatic passage, combined with some of the ideas of quantum control theory. We will illustrate our method with a novel implementation of quantum computing using atoms confined in optical lattices [3,4]. If the number of atoms in each of the potential wells is uncertain, which is one of the problems with this kind of experiment, most of the parameters will have an uncertainty of the order of the parameter itself (e.g., $|\Delta\Omega| \sim \Omega$), which under normal circumstances will give rise to very poor fidelities and even impede the performance of quantum gates. As we will show, using our method not only quantum computation is possible but even very high fidelities could be achieved.

The use of adiabatic passage techniques to implement quantum gates is, of course, not a new idea. In fact, several methods to perform certain quantum gates using Berry phases have been put forward recently [5–7]. Furthermore, adiabatic passage techniques have been proposed as a way of implementing a universal set of holonomies [7], i.e., quantum gates which are carried out by varying certain parameters and whose outcome depends only on geometrical properties of the paths in parameter space [5]. In all these proposals, physical implementations of standard quantum computation have been adapted so that the quantum gates are performed in an adiabatic way giving rise to holonomies. Despite its clear fundamental interest, it is not clear yet if such a novel way of implementing the quantum gates may offer real benefits with respect to the original proposals. In contrast, in our illustrative example adiabatic passage is required to perform quantum gates and therefore it is an essential tool not only to achieve the desired precision but also to build a quantum computer at all.

The outline of the Letter is as follows. First we show how to produce a universal set of gates (Hadamard, phase, and CNOT) starting from two Hamiltonians in which only one parameter is precisely controlled. Then we show that this method eliminates an important obstacle in a particular physical scenario that has been proposed for quantum computation [3], namely, a set of atoms in optical lattices interacting via cold collisions.

Let us consider a set of qubits that can be manipulated according to the single qubit Hamiltonian

$$H_1 = \frac{\Delta}{2}\sigma_z + \frac{\Omega}{2}(\sigma_+ e^{i\varphi} + \sigma_- e^{-i\varphi}) \tag{1}$$

and the two-qubit Hamiltonian

$$H_2 = \tilde{\Delta}|11\rangle\langle 11| + \frac{\tilde{\Omega}}{2}\mathbb{1} \otimes (\sigma_+ e^{i\varphi} + \sigma_- e^{-i\varphi}). \tag{2}$$

As mentioned above, we assume that most of the parameters appearing in these Hamiltonians are basically unknown. In particular, we assume that only φ can be precisely controlled. For the other parameters we assume that (i) they are given by an unknown (single valued) function of some experimentally controllable parameters and (ii) they can be set to zero. For example, we may have $\Omega = f(I)$, where I is a parameter that can be experimentally controlled, and we know only about f that $f(0) = 0$ and that we can reach some value $\Omega_m \equiv f(I_m) \neq 0$ for some I_m [8]. Outside of this, $f(I)$ may change in different experimental realizations. Below we analyze a particular

physical scenario which exactly corresponds to this situation, but we stress that these conditions can be naturally met in more general scenarios. For example, the qubit states $|0\rangle$ and $|1\rangle$ may correspond to two degenerate atomic (ground state) levels which are driven by two lasers of the same frequency and different polarization. The corresponding Hamiltonian is given by (1), where the parameters φ, Ω, Δ describe the relative phase of the lasers, the Rabi frequency, and detuning of the two-photon Raman transition, respectively. The Rabi frequency can be changed by adjusting the intensity of the lasers, and the detuning and the phase difference by using appropriate modulators. In practice, Ω (Δ) can be set to zero very precisely by switching off the lasers (modulators) and φ may be very precisely controlled to any number between 0 and 2π. However, fixing Ω or Δ to a precise value (for example, 23.098 kHz) may be much more challenging.

The idea of obtaining perfect gates with unknown parameters relies on the combination of adiabatic passage techniques [5] and ideas of quantum control [9]. Let us recall the basic idea of adiabatic passage. Suppose we have a Hamiltonian that depends parametrically on a set of parameters, denoted by p, which are changed adiabatically with time along a given trajectory $p(t)$. After a time T, the unitary operator corresponding to the evolution is

$$U(T) = \sum_{\alpha} e^{i(\phi_\alpha + \psi_\alpha)} |\Phi_\alpha[p(T)]\rangle\langle\Phi_\alpha[p(0)]|. \qquad (3)$$

Here, $|\Phi_\alpha(p)\rangle$ are the eigenstates of the Hamiltonian for which the parameters take on the values p. The phase ϕ_α is a dynamical phase that explicitly depends on how the parameters p are changed with time, whereas the phase ψ_α is a purely geometrical phase and depends on the trajectory described in the parameter space. Our basic idea to perform any given gate is first to design the change of the parameters in the Hamiltonians (1) and (2) such that the eigenvectors evolve according to the desired gate, and then to repeat the procedure changing the parameters appropriately in order to cancel the geometric and dynamical phases.

Let us first show how to perform the phase gate $U = e^{i\theta\sigma_z/2}$. We set $\Delta = 0$ for all times. The parameters (Ω, φ) have to be changed as follows [see Fig. 1(a)]:

$$\begin{aligned}(0,0) &\xrightarrow{\text{(i)}} (\Omega_m, 0) \xrightarrow{\text{(ii)}} (\Omega_m, \theta/2) \overset{\text{(iii)}}{\Rightarrow} (\Omega_m, \theta/2 + \pi) \\ &\xrightarrow{\text{(iv)}} (\Omega_m, \theta + \pi) \xrightarrow{\text{(v)}} (0, \theta + \pi). \end{aligned} \qquad (4)$$

Steps (i),(ii) and (iv),(v) are performed adiabatically and require a time T. The double arrow of step (iii) indicates a sudden change of parameters. Note that $\Omega(0) = \Omega(2T) = 0$, $\Omega(t) = \Omega(2T - t)$, and $\varphi(t) = \pi + \theta - \varphi(2T - t)$, which does not require the knowledge of the function f but implies a precise control of the phase. A simple analysis shows that (i)–(v) achieve the desired transformation $|0\rangle \to e^{i\theta/2}|0\rangle$, $|1\rangle \to e^{-i\theta/2}|1\rangle$. Note also that the dynamical and geometrical phases acquired in the adiabatic processes (i)–(v) cancel out.

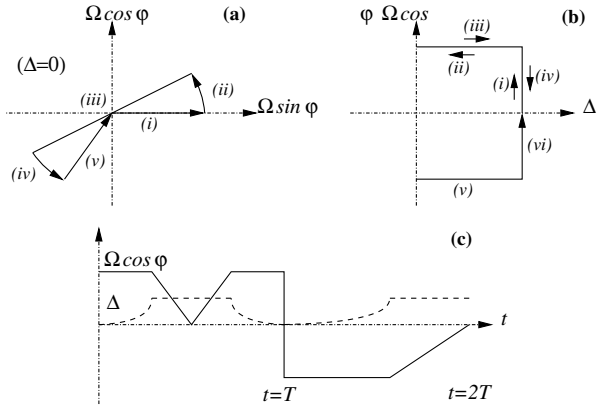

FIG. 1. Schema of how the parameters of Hamiltonian (1) have to be changed in order to perform a phase gate (a) and Hadamard gate (b). In (c) we show a possible evolution of the parameters Δ and Ω for the Hadamard gate. The shape of Δ emphasizes that the actual dependence is unimportant.

The Hadamard gate can be performed in a similar fashion. In the space of $[\Delta, \Omega_x = \Omega\cos(\varphi)]$, the protocol is

$$\begin{aligned}(0, \Omega_m) &\xrightarrow{\text{(i)}} (\Delta_m, \Omega_m) \xrightarrow{\text{(ii)}} (\Delta_m, 0) \xrightarrow{\text{(iii)}} (\Delta_m, \Omega_m) \xrightarrow{\text{(iv)}} (0, \Omega_m) \\ &\overset{\text{(v)}}{\Rightarrow} (0, -\Omega_m) \xrightarrow{\text{(vi)}} (\Delta, -\Omega_m) \xrightarrow{\text{(vii)}} (\Delta, 0), \end{aligned} \qquad (5)$$

as shown in Figs. 1(b) and 1(c). In order to avoid the dynamical phases, we have to make sure that steps (i)–(v) are run in half the time as (vi)–(vii). More precisely, if $t < T$, we must ensure that $\Delta(t) = \Delta(T - t)$, $\Omega_x(t) = \Omega_x(T - t)$, $\Delta(T + t) = \Delta(t/2)$, and $\Omega_x(T + t) = \Omega_x(t/2)$. With this requisite we get $(1/\sqrt{2})(|0\rangle + |1\rangle) \to |0\rangle$, $(1/\sqrt{2})(|0\rangle - |1\rangle) \to -|1\rangle$. Again, the whole procedure does not require us to know Ω or Δ, but rather to control the evolution of the experimental parameters which determine them.

The CNOT gate requires the combination of two two-qubit processes using H_2 and one local gate. The first process involves changing the parameters $[\tilde{\Delta}, \tilde{\Omega}_x = \tilde{\Omega}\cos(\varphi)]$ of Eq. (2) according to

$$\begin{aligned}(\tilde{\Delta}_m, 0) &\xrightarrow{\text{(i)}} (\tilde{\Delta}_m, \tilde{\Omega}_m) \xrightarrow{\text{(ii)}} (0, \tilde{\Omega}_m) \overset{\text{(iii)}}{\Rightarrow} (0, -\tilde{\Omega}_m) \\ &\xrightarrow{\text{(iv)}} (\tilde{\Delta}_m, -\tilde{\Omega}_m) \xrightarrow{\text{(v)}} (\tilde{\Delta}_m, 0). \end{aligned} \qquad (6)$$

This procedure gives rise to the transformation

$$U_1 = |0\rangle\langle 0| \otimes \mathbb{1} + e^{i\xi}|1\rangle\langle 1| \otimes i\sigma_y, \qquad (7)$$

where $\xi = \int_0^T \delta(t)dt$ is an unknown dynamical phase. The second operation required is a NOT on the first qubit $U_2 = (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes \mathbb{1}$. Finally, if $\tilde{\Delta}^{(1)}(t)$ denotes the evolution of $\tilde{\Delta}$ in Eq. (6), we need to follow a path such that $\tilde{\Delta}^{(3)}(t) = \tilde{\Delta}^{(1)}(t)$, $\tilde{\Omega}^{(3)}(t) = 0$. If the timing

is correct, we achieve $U_3 = (|0\rangle\langle 0| + e^{i\xi}|1\rangle\langle 1|) \otimes \mathbb{1}$. Everything combined gives us the CNOT up to a global unimportant phase $U_{\mathrm{CNOT}} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes i\sigma_y = e^{-i\xi} U_2 U_3 U_2 U_1$.

In Fig. 2(a) we illustrate the performance of our method, as well as its sensitivity against nonadiabatic processes. As a figure of merit we have chosen the gate fidelity [10] $\mathcal{F} = |\mathrm{Tr}\{U_{\mathrm{ideal}}^{\dagger} U_{\mathrm{real}}\}|^2/d^2$, where d is the dimensionality of the space (2 for local gates, 4 for two-qubit gates), U_{ideal} is the gate that we wish to produce and U_{real} is the actual operation performed. As expected, for fixed parameters $\{\Delta_m/\Omega_m = \tilde{\Delta}_m/\tilde{\Omega}_m = 1/10, \varphi_m = \pi/4\}$, the adiabatic theorem applies when the processes are performed with a sufficiently slow speed. Typically a time $T \sim 300/\Omega_m, 300/\tilde{\Omega}_m$ is required for the desired fidelity $\mathcal{F} = 1 - 10^{-4}$ [Fig. 2(a).]

Let us now consider a set of bosonic atoms confined in a periodic optical lattice at sufficiently low temperature (such that only the first Bloch band is occupied). The atoms have two relevant internal (ground) levels, $|a\rangle$ and $|b\rangle$, in which the qubit is stored. This setup has been considered in Ref. [3] where it has been shown how single quantum gates can be realized using lasers and two-qubit gates by displacing the atoms that are in a particular internal state to the next neighbor location. The basic ingredients of such a proposal have been recently realized experimentally [11]. However, in this and all other schemes so far [12] it is assumed that there is a single atom per lattice site since otherwise even the concept of qubit is no longer valid. In present experiments, in which the optical lattice is loaded with a Bose-Einstein condensate [13,14], this is not the case, since zero temperature is required and the number of atoms must be identical to the number of lattice sites. We show now a novel implementation in which, with the help of the methods presented above, one overcomes this problem.

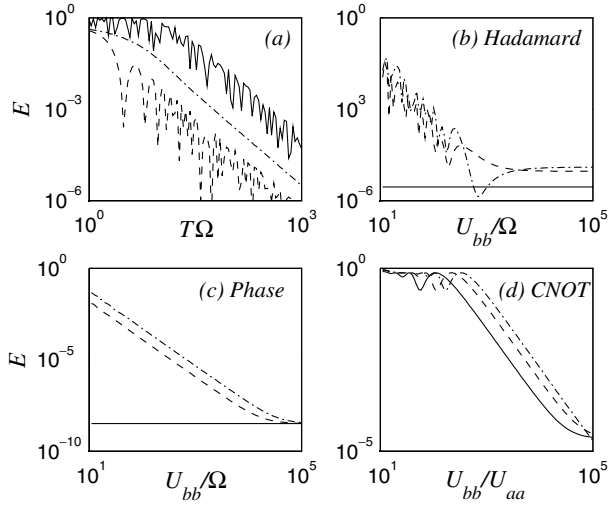

FIG. 2. Log-log plot of the gate error, $E = 1 - \mathcal{F}$, for different local and nonlocal gates. (a) Realization based on Hamiltonians (1) and (2), with fixed parameters and varying time. We plot the error for the Hadamard (solid line), phase (dashed line), and CNOT gates (dotted line). (b)–(d) Errors for our proposal with atomic ensembles. In (b) and (c) we plot realizations for $n = 1$ (solid line) and $n = 3, 5$ (dashed line) atoms per lattice site. In (d) we plot simulations with a difference of atoms of $|n - m| = 0, 1, 2$ between both wells (solid, dashed, and dotted lines). All parameters except U_{bb} are fixed to values given in the text. The curves (a),(b) appear random due to the sampling of data.

For us *a qubit will be formed by an aggregate of atoms at some lattice site*. The number of atoms forming each qubit is completely unknown. The only requirement is that there is at least one atom per site [15]. We denote by n_k the number of atoms in the kth well and identify the states of the corresponding qubit as

$$|0\rangle_k = \frac{1}{\sqrt{n!}} a_k^{\dagger n_k}|\mathrm{vac}\rangle, \qquad |1\rangle_k = \frac{1}{\sqrt{n}} b_k^{\dagger} a_k |0\rangle_k, \tag{8}$$

where $a_k^{\dagger}$ ($b_k^{\dagger}$) are the creation operators for one atom in levels $|a\rangle$ and $|b\rangle$, respectively. The quantum gates will be realized using lasers, switching the tunneling between neighboring sites, and using the atom-atom interaction.

In the absence of any external field, the Hamiltonian describing our system is

$$H = \sum_k \left[-J_k^{(b)} (b_{k+1}^{\dagger} b_k + b_k^{\dagger} b_{k+1}) + \frac{1}{2} U_{bb} b_k^{\dagger} b_k^{\dagger} b_k b_k \right]. \tag{9}$$

Here, U_{bb} and $J_k^{(b)}$ describe the interactions between and the tunneling of atoms in state $|b\rangle$. We assume that $J_k^{(b)}$ can be set to zero and increased by adjusting the intensities of the lasers which create the optical lattice. We have assumed that the atoms in state $|a\rangle$ do not interact at all and do not hop, something which may be achieved by tuning the scattering lengths and the optical lattice. Both restrictions will be relaxed later on. The Hamiltonian (9) possesses a very important property when all $J_k^{(b)} = 0$; namely, it has no effect on the computational basis (i.e., $H|\Psi\rangle = 0$ for all states $|\Psi\rangle$ in the Hilbert space generated by the qubits). Otherwise, it would produce a nontrivial evolution that would spoil the computation.

We show now how a single qubit gate on qubit k can be realized using lasers. First, during the whole operation we set $J_k^{(b)} = 0$ in order to avoid hopping. The laser interaction is described by the Hamiltonian

$$H_{\mathrm{las}}^{(k)} = \frac{\Delta_k}{2} (a_k^{\dagger} a_k - b_k^{\dagger} b_k) + \frac{\Omega_k}{2} (e^{i\varphi} a_k^{\dagger} b_k + e^{-i\varphi} b_k^{\dagger} a_k). \tag{10}$$

For $U_{bb} \gg |\Delta_k|, |\Omega_k|$, we can replace (10) by an effective Hamiltonian which resembles (1), with $\Delta = \Delta_k$ and $\Omega = \Omega_k \sqrt{n_k - 1}$. Thus, using the methods exposed above we can achieve the Hadamard and phase gates with a high precision, even though the coupling between the bosonic ensemble and the light depends on the number of atoms.

For the realization of the two-qubit Hamiltonian (2) we need to combine several elements. First of all we need the Raman coupling of Eq. (10) to operate on the second well. Second, we need to tilt the lattice using an electric field, $H_{\rm tilt} = \sum_k kg(a_k^\dagger a_k + b_k^\dagger b_k)$. And finally we must allow virtual hopping of atoms of type $|b\rangle$ ($J^{(b)} \ll |U_{bb} - g|$). After adiabatic elimination we find that the effective Hamiltonian depends on the number of particles in the second site, n_2,

$$H_2^{\rm eff} = \frac{J_b^2}{g - U_{bb}} |11\rangle\langle 11| + \sqrt{n_2 - 1}\tilde{\Omega}\, \mathbb{1} \otimes \sigma_x. \qquad (11)$$

The identification with Eq. (2) is evident, and once more the use of adiabatic passage will produce gates which are independent of the number of particles.

We have studied the different sources of error which may affect our proposal: (i) U_{bb} is finite and (ii) atoms in state $|a\rangle$ may hop and interact. These last phenomena are described by additional contributions to Eq. (9) which are of the form $J_k^{(a)}(a_k^\dagger a_{k+1} + a_{k+1}^\dagger a_k)$, $U_{aa} a_k^\dagger a_k^\dagger a_k a_k$, and $U_{ab} a_k^\dagger b_k^\dagger a_k b_k$. The consequences of both imperfections are (i) more than one atom per well can be excited, (ii) the occupation numbers may change due to hopping of atoms, and (iii) by means of virtual transitions the effective Hamiltonian differs from (1) and (2). The effects (i),(ii) are eliminated if $(\Omega/U_{bb})^2 \ll 1$ and $[J_k^{(a)}/U_{bb}]^2 \ll 1$. We may analyze the remaining errors with a perturbative study of the Hamiltonians (10) and (9) plus the terms $(J^{(a)}, U_{ab}, U_{aa})$ that we did not consider before. In Eq. (10), the virtual excitation of two atoms increments the parameter Δ by an unknown amount, $\Delta_{\rm eff} \sim \Delta + 2\Omega^2 n_k/(\Delta + U_{ab} - U_{bb})$. If $U_{ab} \ll U_{bb}$ and $\Omega^2 n_k T/U_{bb} \ll 1$, this shift may be neglected. In the two-qubit gates the energy shifts are instead due to virtual hopping of all types of atoms. They are of the order of $\max[J^{(b)}, J^{(a)}]^2/g^2 \sim J^2/U_{bb}$, and for $J^2 T/U_{bb} \ll 1$ they also may be neglected.

To quantitatively determine the influence of these errors we have simulated the evolution of two atomic ensembles with an effective Hamiltonian which results of applying second order perturbation theory to Eq. (9), and which takes into account all important processes. The results are shown in Figs. 2(c) and 2(d). For the two-qubit gate we have assumed $U_{aa} = U_{ab}$, $J^{(a)} = J^{(b)} = J$, $J_m = 0.05 U_{ab}$, $\Omega_m = J_m^2/10$, $g = U_{bb} + U_{ab}/2$, and operation time $T = 200/\Omega_m$, while changing the ratio U_{bb}/U_{ab} and the populations of the wells. For the local gates we have assumed $\Delta_m = 1$, $\Omega_m = J/10$, and different occupation numbers n_k, and we have also changed U_{bb}/Ω_m.

We extract several conclusions. First, the stronger the interaction between atoms in state $|b\rangle$, the smaller the energy shifts. Typically, a ratio $U_{bb} = 10^4 U_{ab}$ is required to make $\mathcal{F} = 1$–10^{-4}. Second, the larger the number of atoms per well, the poorer the fidelity of the local gates [Figs. 2(b) and 2(c)]. And finally, as Fig. 2(d) shows, the population imbalance between wells influences very little the fidelity of the two-qubit gate.

Finally, we mention the effect of fluctuations of the control parameters. By expanding our Hamiltonian as $H = H_{\rm ad}(t) + \Delta I(t) H_{\rm pert}$, where $\Delta I(t)$ represents the deviation from our adiabatic path, and writing the evolution operator in the form $U(t,0) = U_{id}(t,0)W(t)$, with $i\frac{d}{dt}U_{\rm ad} = H_{\rm ad}U_{\rm ad}$, one may prove that the errors due to random quick fluctuations are $E = 1 - |{\rm Tr}\{W(t)\}|^2 = 2\int \Delta I(t) {\rm Tr}\{Q\} + \mathcal{O}(\Delta I^2)$. ${\rm Tr}\{Q\}$ is some constant, and thanks to the randomness of $\Delta I(t)$, the errors are at most quadratic in the perturbation.

In this work we have shown that it is possible to perform quantum computation even when the constants in the governing Hamiltonians are unknown, by means of adiabatic passage with one-qubit and two-qubit Hamiltonians. Based on this, we have proposed a scheme for quantum computing with cold atoms in a tunable optical lattice, which works even when the number of atoms per lattice site is unknown. Note that these ideas also apply to other setups, such as the microtraps realized in Ref. [16].

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