Light Sheets and Bekenstein's Entropy Bound

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From the covariant bound on the entropy of partial light sheets, we derive a version of Bekenstein's bound: $S/M \leq \pi x/\hbar$, where S, M, and x are the entropy, total mass, and width of any isolated, weakly gravitating system. Because x can be measured along any spatial direction, the bound becomes unexpectedly tight in thin systems. Our result completes the identification of older entropy bounds as special cases of the covariant bound. Thus, light sheets exhibit a connection between information and geometry far more general, but in no respect weaker, than that initially revealed by black hole thermodynamics.

Entropy bounds have undergone a remarkable transformation from a corollary to a candidate for a first principle [1]. After proposing the generalized second law of thermodynamics (GSL) [2,3]—that the sum of black hole entropy and ordinary matter entropy never decreases—Bekenstein argued that its validity necessitates a model-independent bound [4,5] on the entropy S of weakly gravitating systems:

$$S \leq \pi M d/\hbar, \tag{1}$$

where M is the total gravitating energy, and d is the linear size of the system, defined to be the diameter of the smallest sphere that fits around the system. This inequality is obtained by considering the classical absorption of the system by a large black hole; it does not depend on the dimension of spacetime [6]. Bekenstein's bound is remarkably tight (consider, for example, a massive particle in a box the size of its Compton wavelength). It has appeared in discussions ranging from information technology to quantum gravity. Since $M \ll d/4G$ for a weakly gravitating system, it also implies the "spherical entropy bound,"

$$S \leq A_{cs}/4G\hbar. \tag{2}$$

Here A_{cs} is the area of the circumscribing sphere.

Though confined to weak gravity, 't Hooft [7] and Susskind [8] ascribed fundamental significance to Eq. (2), claiming that it reflects a nonextensivity of the number of degrees of freedom in nature. This eventually prompted the conjecture of a more general bound, the covariant entropy bound [9]. Empirically, this bound has been found to hold in large classes of examples, including systems in which gravity is the dominant force. Meanwhile, no violation has been observed, nor have any theoretical counterexamples been constructed from a realistic effective theory of matter and gravitation.

Although the covariant bound does not conflict with the phenomenology of our present models, it cannot be derived from known principles. It may be interpreted as

an unexplained pattern in nature, betraying a fundamental relation between information and spacetime geometry. Then the bound must eventually be explained by a unified theory of gravity, matter, and quantum mechanics. In the mean time, it should be regarded as providing important hints about such a theory.

We are thus motivated to consider the covariant entropy bound primary and to try to derive other laws of physics from it. As we will discuss shortly, the bound has already been shown to imply the GSL, as well as older, more specialized entropy bounds. However, the oldest (and, for weakly gravitating systems, tightest) bound of all, Bekenstein's bound, is an exception. It has not previously been identified as a special case of the covariant bound.

The main purpose of this note is to fill this gap. We will use the covariant bound, in the stronger form of Ref. [10], to derive an inequality of the type introduced by Bekenstein, Eq. (1). Our result will be obtained directly, without use of the GSL. Thus we circumvent the continued debate of whether Bekenstein's bound is really necessary for the GSL [11,12].

Let us briefly review the covariant bound and its logical relation to the GSL and to the spherical bound, Eq. (2). Given any open or closed spatial surface B at a fixed instant of time, one can always construct at least two light sheets. A light sheet of B is a null hypersurface generated by nonexpanding light rays (i.e., null geodesics) which emanate from B orthogonally [9]. For example, for a spherical surface in Minkowski space, the two light sheets will be the two light cones ending on B.

The covariant entropy bound [9] claims that the entropy S of the matter on any light sheet L of B is bounded by the surface area $A(B)$:

$$S[L(B)] \leq A(B)/4G\hbar, \tag{3}$$

where G is Newton's constant. (We set Boltzmann's constant and the speed of light to 1.) The entropy S refers to the total entropy of all matter systems that are "seen" by

the light rays generating L (systems whose world volume is fully intersected by L).

Let B be a complete cross section of the horizon of a black hole. Then its past-directed ingoing light sheet intersects with all the matter systems that collapsed to form the black hole [10]. Moreover, $A(B)/4G\hbar$ in this case represents the Bekenstein-Hawking entropy of the black hole. The bound thus guarantees that the black hole entropy exceeds the matter entropy lost to an outside observer. That is, the GSL is upheld when a black hole forms.

The GSL should also hold when a matter system falls into an existing black hole. In that case it requires that the black hole horizon area increases enough so that the additional Bekenstein-Hawking entropy compensates for the loss of matter entropy: $S \leq \Delta A_{\text{horizon}}/4G\hbar$. In the form of Eq. (3), the covariant entropy bound does not imply this relation. This prompted Flanagan, Marolf, and Wald [10] to propose a stronger formulation, the "generalized" covariant entropy bound (GCEB),

$$S[L(B; B')] \leq \frac{A(B) - A'(B')}{4G\hbar}. \qquad (4)$$

Here A' is the area of any cross-sectional surface B' on the light sheet L of B. S denotes the entropy of matter systems found on the portion of L between B and B' [13].

Put differently, in constructing L, we are at liberty to follow each light ray until it intersects with neighboring light rays. (At these caustic points the light rays begin to diverge, and the nonexpansion condition becomes violated.) But nothing forces us to follow each light ray to the bitter end. We may construct a *partial light sheet* by terminating L before caustics are reached. Then the endpoints of the light rays will span a nonzero area A'. It is natural to expect that the inequality (3) can be tightened in this case, because we are not including in S all the matter systems that could have been reached by the light rays. Equation (4) improves the bound accordingly.

The GCEB does imply the GSL for all processes involving black holes, including the absorption of a matter system by an existing black hole [10]. It also, of course, implies the weaker form of the covariant bound (3), which in turn implies the spherical entropy bound (2) in weakly gravitating regions [9].

To derive Bekenstein's bound from the GCEB, we wish to apply Eq. (4) to an isolated, weakly gravitating matter system. The basic idea of our proof is to "x ray" the system. Because matter bends light, initially parallel geodesics will arrive on the "image plate" slightly contracted. The resulting area difference, which bounds the system's entropy, will be expressed as the product of the mass and the width of the system.

We make the following *assumptions*: (i) The stress tensor T_{ab} has support only in a spatially compact region, the world volume W of the matter system. (ii) Gravity is weak. Specifically: (ii.1) The metric is approximately flat: $g_{ab} = \eta_{ab} + \delta g_{ab}$, with $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ and

$|\delta g_{ab}| \ll 1$. (ii.2) Any part of the system is much smaller than any (averaged) curvature scale it produces: $\langle R_{abcd} \rangle_\ell \ll \ell^{-2}$, where $\langle R_{abcd} \rangle_\ell$ is the average Riemann tensor along a distance ℓ. It is believed that all physical matter (at least when suitably averaged) satisfies the null and causal energy conditions. These conditions may also be needed for the validity of the GCEB, which however is being assumed here in any case. To derive Bekenstein's bound we shall require only the null energy condition: (iii) $T_{ab}k^a k^b \geq 0$ for any null vector k^a.

We begin with some definitions valid at zeroth order in δg. Cartesian coordinates x^μ ($\mu = 0, \ldots, D - 1$) cover the spacetime. The corresponding vector fields $\partial/\partial x^\mu$ define an orthonormal frame at every point, which we take to be a rest frame of W for convenience. (The remaining choice of spatial orientation will be exploited later.) The curves

$$x^0 = x^1; \qquad (x^2, \ldots, x^{D-1}) \text{ arbitrary constants}, \qquad (5)$$

describe a set of parallel light rays traveling in the x^1 direction (see Fig. 1). More precisely, they define a null geodesic congruence L, with affine parameter x^1 and everywhere vanishing expansion. We will be interested only in the intersection of the hypersurface L with the world volume W of the matter system. Let $\tilde{B}_+$ ($\tilde{B}_-$) be the set of the first (last) points of each light ray in W. They form $(D - 2)$-dimensional spatial surfaces characterized by functions $x^1_\pm(x^2, \ldots, x^{D-1})$, with a finite range for $(x^2, \ldots, x^{D-1})$. (Connectedness is not necessary for this proof.) All spatial sections of a light sheet are surface orthogonal to the generating light rays. Hence, $L \cap W$ is a partial light sheet with initial and final surfaces $\tilde{B}_\pm$. At zeroth order they have equal area.

In the exact metric, we may use the same coordinates. Generically, however, the hypersurface L as defined by Eq. (5) will be neither null nor made of geodesics, nor is there a sense of strictly nonpositive expansion. All of these qualitative conditions must hold for L to be a light sheet; otherwise the GCEB cannot be applied. Hence we must adjust L slightly. We will define two light sheets, $L_\pm$, both of which limit to L as $\delta g \rightarrow 0$.
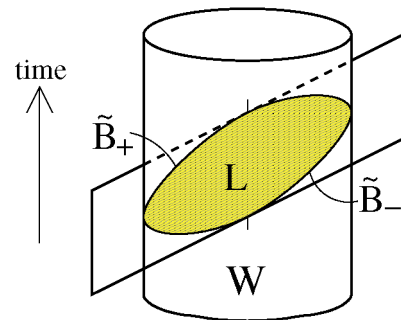

FIG. 1 (color online). Matter system W, light sheet L, entry surface $\tilde{B}_+$, and exit surface $\tilde{B}_-$. At first order in δg, the bending of light leads to a small area difference between entry and exit surfaces, which bounds the entropy of W.

Consider the future- and W-directed light rays orthogonal to $\tilde{B}_+$. Because gravity is weak, their expansion will be very small (compared to the inverse width of W). But it will not vanish exactly, and it need not be of definite sign. However, $\tilde{B}_+$ is embedded with codimension 1 in the boundary of W, ∂W. Thus there exists a small (non-unique) deformation of $\tilde{B}_+$ within ∂W, the surface B_+, whose orthogonal null geodesics have initially vanishing expansion *to all orders* [14]. Assumption (iii) ensures that θ will not increase away from $\tilde{B}_+$ [15], and (ii.2) excludes the possibility that the light rays intersect within W [16]. Hence, the light rays generate a light sheet L_+ that captures all of the matter system. Let A_+ be the area of B_+, and let A'_+ be the area of the surface B'_+ spanned by the same light rays when they (last) leave the system W.

Similarly, $\tilde{B}_-$ can be deformed to a surface B_- of exactly vanishing expansion. This defines a second, slightly different light sheet L_- with initial and final areas A_- and A'_-. The light sheets $L_\pm$ have opposite directions of contraction, roughly $\pm x^1$. We will be interested in the total change of the cross-sectional area as each light sheet traverses W: $\Delta A_+ \equiv A_+ - A'_+$ and $\Delta A_- \equiv A_- - A'_-$.

Let S be the entropy of the matter system, i.e., the logarithm of the number of independent quantum states accessible to any system of total mass M occupying the world volume W in a neighborhood of $L_\pm$. As each light sheet fully contains the matter system, the GCEB implies that $S \leq \Delta A_+/4G\hbar$ and also that $S \leq \Delta A_-/4G\hbar$. Therefore,

$$S \leq \frac{\Delta A_+ + \Delta A_-}{8G\hbar}. \tag{6}$$

To calculate $\Delta A_\pm$ to leading order, we may continue using $(x^2, \ldots, x^{D-1})$ to label the light rays in $L_\pm$. We may approximate the affine parameter along each ray by $\pm x^1$ and the vector field tangent to the light rays by

$$\pm k^a = \left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right)^a. \tag{7}$$

Let $\mathcal{A}_\pm$ be the cross-sectional area spanned by the light ray $(x^2, \ldots, x^{D-1})$ and its infinitesimally neighboring light rays in the light sheet $L_\pm$ at the (affine) position x^1. At each point on each of the two light sheets, the expansion $\theta_\pm$ is given by the trace of the null extrinsic curvature [1]. Equivalently, it is the logarithmic derivative of $\mathcal{A}_\pm$ with respect to the affine parameter $\pm x^1$:

$$\theta_\pm(x^1; x^2, \ldots, x^{D-1}) = \pm \frac{d\mathcal{A}_\pm/dx^1}{\mathcal{A}_\pm}. \tag{8}$$

Raychaudhuri's equation,

$$\frac{d\theta}{d(\pm x^1)} = -\frac{1}{2}\theta_\pm^2 - \sigma_\pm^2 - R_{ab}k^a k^b, \tag{9}$$

describes how the expansion changes along a light ray. Here R_{ab} is the Ricci tensor. There is no twist term

because the light sheets are surface orthogonal [15]. The expansion and shear terms, θ^2 and σ^2, are of higher order than the stress term and can be neglected. In this approximation one can integrate Eqs. (8) and (9) for each light ray:

$$\mathcal{A}_+(x^1) = \mathcal{A}_+(x^1_+) \exp \int_{x^1_+}^{x^1} d\hat{x}^1 \theta_+(\hat{x}^1) \tag{10}$$

$$= \mathcal{A}_+(x^1_+)\left[1 - \int_{x^1_+}^{x^1} d\hat{x}^1 \int_{x^1_+}^{\hat{x}^1} d\hat{\hat{x}}^1 R_{ab}k^a k^b\right]. \tag{11}$$

For $x^1 = x^1_-$, the curvature term yields the fractional change in each area element $dx^2, \ldots, dx^{D-1}$. By assumption (ii.2), this term will be small compared to unity. The area change can be integrated to obtain

$$\Delta A_+ = 8\pi G \int dx^2 \cdots dx^{D-1} \int_{x^1_+}^{x^1_-} d\hat{x}^1 \int_{x^1_+}^{\hat{x}^1} d\hat{\hat{x}}^1 T_{ab}k^a k^b. \tag{12}$$

After adding the analogous expression for the light sheet L_-, the integrals factorize, and Eq. (6) becomes

$$S \leq \frac{\pi}{\hbar} \int dx^2 \cdots dx^{D-1} \, \Delta x^1 \int_{x^1_+}^{x^1_-} dx^1 T_{ab}k^a k^b. \tag{13}$$

To continue the inequality, we replace the local width of the system, $\Delta x^1 \equiv x^1_- - x^1_+$, by its largest value over $(x^2, \ldots, x^{D-1})$, x. (For convex systems, x is the separation of two planes orthogonal to x^1, which "clamp" W; but generally, it can be smaller than that.) This yields

$$S \leq \pi P_b k^b x/\hbar, \tag{14}$$

where $P_b \equiv \int dx^1 \cdots dx^{D-1} T_{ab}k^a$. Note that P_b is a correctly normalized integral of the conserved tensor T_{ab} over a null hypersurface [see Eq. (7) and, e.g., Appendix B.2 in Ref. [15]]. Since T_{ab} vanishes outside W, the hypersurface of integration can be extended to spatial infinity without affecting the value of P_b. Hence the time component of P_b is the total energy, and the (negative) spatial components are the Arnowitt-Deser-Misner momenta. In a rest frame, the momenta vanish by definition, and P_0 is equal to the system's total ("rest") mass M. We thus obtain a "generalized Bekenstein bound,"

$$S \leq \pi M x/\hbar. \tag{15}$$

Our result is somewhat stronger than the original Bekenstein bound, Eq. (1), because of our definition of the relevant length scale, x. Bekenstein advocated using the largest scale of the system, the circumferential diameter d. Our argument, however, allows us to use the smallest dimension. For example, if the system is rectangular with sides of length $a < b < c$, we are free to align the x^1 axis with the shortest edge, so that $x = a$. For more general shapes, the strength of Eq. (15) is optimized as follows. Find the greatest width of the system, $x(\Omega)$, for every orientation Ω of the system relative to the x^1 axis; then choose the particular orientation $\Omega_{\min}$ that yields the

smallest such greatest width, $x(\Omega_{\min})$. If the shape of the system is time dependent, then x can be minimized not only by judicious rotations, but also by time translations of W relative to L [17]. Independently of the shape of the system, $x \leq d$ for all Ω, and, in particular, for $\Omega_{\min}$. Hence Eq. (15) implies Eq. (1). For systems with highly unequal dimensions, such as a very thin box, $x \ll d$. In this case Eq. (15) is much stronger than Eq. (1).

The assumptions we stated earlier characterize the regime in which the generalized Bekenstein bound can be applied. Our construction will not go through unless the system is compact and isolated, so that initial and final surfaces of a suitable light sheet can be constructed. The weakness of gravity ensures that the light sheet area decrease is small and that it is given by the product of a (well-defined) width and mass.

Thus, our derivation does not give licence to all interpretations the Bekenstein bound has received. For example, we do not find support for its application to a closed universe. Let S be the entropy of the quantum fields on a spatial three sphere of diameter d at total energy M. (These quantities are well defined in the absence of gravity, $G = 0$.) In this case the system occupies a geometry which is intrinsically curved. Unlike an isolated system in flat space, it cannot be fully covered by a partial light sheet. Hence, the covariant bound does not imply Bekenstein's bound in this case. Indeed, violations of Eq. (1) were found for supersymmetric conformal field theories on spatial spheres of various dimensions [18].

There is no evidence that the original Bekenstein bound is violated by any complete, isolated, weakly gravitating system that can actually be constructed in nature [12,19]. It also appears to be reasonably tight, in that realistic matter can come within an order of magnitude of saturating the bound [5]. But the generalized Bekenstein bound faces challenges to which the original was immune. Testing Eq. (15) is important both in its own right and as a simple check of the GCEB that obviates the computation of geodesics. Detailed examples will be presented elsewhere.

We close on a speculative note. Gravity plays a central role in our derivation of Bekenstein's bound. We combined the GCEB, a conjecture involving the Planck area $G\hbar$, with classical equations involving G. But in due course G dropped out, leaving only $\hbar$ in the final result. Indeed, Bekenstein's bound can be tested entirely within quantum field theory, apparently without any use of the laws of gravity [5]. This remarkable fact suggests a novel perspective on the connection between gravity and quantum mechanics. Note that for systems with small numbers of quanta ($S \approx 1$), Bekenstein's bound can be seen to *require* nonvanishing commutators between conjugate variables, as they prevent Mx from becoming much smaller than $\hbar$. One is tempted to propose that at least one of the principles of quantum mechanics implicitly used in any veri-

fication of Bekenstein's bound will ultimately be recognized as a consequence of Bekenstein's bound and thus of the covariant entropy bound and of the holographic relation it establishes between information and geometry.

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[13] The GCEB has been proven from a set of phenomenologically motivated assumptions [10], eliminating many possible counterexamples. These assumptions, however, are not fundamental. Unlike the GCEB itself, they require the treatment of entropy as a fluid, which cannot be appropriate at all scales. Moreover, their plausibility is tied to the thermodynamic limit. To avoid confusion, we stress that we will be assuming the GCEB axiomatically. The assumptions and theorems of Ref. [10] do not play any role here.
[14] B_+ is constructed by matching the trace of its extrinsic curvature in ∂W to that of ∂W in the spacetime. This prescription is not overconstrained, as it is analogous to the construction of a minimal surface.
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[16] Strictly, this is demonstrated by Eq. (11) below.
[17] Obviously, boosts, rotations, and translations can change the physical setup only when applied either to L or to W alone. Of these operations, only rotations and time translations are useful for minimizing the bound. Spatial translations are either trivial or equivalent to time translations. Boosting W is equivalent to a rotation of W followed by a boost in the x^1 direction. The latter operation is actually trivial because L is invariant under such boosts. Indeed, Δx^1 scales inversely with $P_b k^b$ under x^1 boosts of W, so that one invariably obtains the product of the rest frame quantities x and M.
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