

Lattice Models with $\mathcal{N} = 2$ Supersymmetry

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We introduce lattice models with explicit $\mathcal{N} = 2$ supersymmetry. In these interacting models, the supersymmetry generators Q^\pm yield the Hamiltonian $H = \{Q^+, Q^-\}$ on any graph. The degrees of freedom can be described as either fermions with hard cores, or as quantum dimers; the Hamiltonian of our simplest model contains a hopping term and a repulsive potential. We analyze these models using conformal field theory, the Bethe ansatz, and cohomology. The simplest model provides a manifestly supersymmetric lattice regulator for the supersymmetric point of the massless $(1+1)$ -dimensional Thirring (Luttinger) model. Generalizations include a quantum monomer-dimer model on a two-leg ladder.

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Supersymmetry is an exceptionally powerful theoretical tool. It can allow exact computations in field theory and string theory, even when interactions are strong. However, since only certain quantities can be computed nonperturbatively, a lattice formulation of supersymmetric field theories would be useful. Efforts to achieve this aim have concentrated mainly on taking a known field theory and discretizing it [1]. Since the supersymmetry algebra involves the Poincaré algebra, putting it on a lattice automatically breaks at least some of the supersymmetries. The difficulty is in ensuring that the supersymmetry is maintained in the continuum limit.

In this Letter, we approach the problem in a different way. We introduce simple lattice models with $\mathcal{N} = 2$ supersymmetry, describing interacting fermions and quantum monomer-dimer systems. We show that the continuum limit of the simplest of these models is a well-known $(1+1)$ -dimensional quantum field theory with $\mathcal{N} = (2, 2)$ superconformal symmetry. We also note the close connections with models of current interest in the study of strongly correlated electrons, and show how supersymmetry can be useful in studying their properties. There are two key questions we will attempt to answer. The first is: What properties can be computed exactly using the supersymmetry? The second is: Which (if any) field theories describe the models in the continuum limit?

Our definition of $\mathcal{N} = 2$ supersymmetry is that the Hamiltonian H is built from two nilpotent fermionic generators denoted Q^+ and $Q^- = (Q^+)^\dagger$ [2]. It is

$$H = \{Q^+, Q^-\}. \quad (1)$$

The fact that Q^+ and Q^- commute with H follows from the nilpotency $(Q^+)^2 = (Q^-)^2 = 0$. Models with $\mathcal{N} = 2$ supersymmetry also have a fermion-number symmetry generated by F with $[F, Q^\pm] = \pm Q^\pm$. We shall show how, in at least some cases, this lattice supersymmetry extends to a space-time super(conformal) symmetry in the field

theory describing the continuum limit. Lattice models with a symmetry involving fermionic generators, such as the t - J model at $J = \pm 2t$, are often called “supersymmetric” in the condensed-matter literature, but do not have a Hamiltonian of the form of Eq. (1).

The models we introduce can be defined on any lattice (or actually, any graph) in any dimension. The simplest model involves a single species of fermion c_i , placed at any site i of the lattice. The fermion obeys the usual anticommutator $\{c_i, c_j^\dagger\} = \delta_{ij}$, and the operator $F = \sum_i c_i^\dagger c_i$ counts the number of fermions. We impose the restriction that the fermions have hard cores, meaning that fermions are not allowed on neighboring sites. A hard-core fermion is created by $c_i^\dagger \mathcal{P}_{\langle i \rangle}$, where the projection operator $\mathcal{P}_{\langle i \rangle}$ requires all sites neighboring i to be empty:

$$\mathcal{P}_{\langle i \rangle} = \prod_{j \text{ next to } i} (1 - c_j^\dagger c_j). \quad (2)$$

The supersymmetry operators are defined by

$$Q^+ = \sum_i c_i^\dagger \mathcal{P}_{\langle i \rangle}, \quad Q^- = \sum_i c_i \mathcal{P}_{\langle i \rangle}. \quad (3)$$

It is easy to verify that $(Q^+)^2 = (Q^-)^2 = 0$. The Hamiltonian is therefore

$$H = \sum_i \sum_{j \text{ next to } i} \mathcal{P}_{\langle i \rangle} c_i^\dagger c_j \mathcal{P}_{\langle j \rangle} + \sum_i \mathcal{P}_{\langle i \rangle}. \quad (4)$$

The first term in the Hamiltonian allows fermions to hop to neighboring sites, with the projectors maintaining the hard-core repulsion. The second term favors having more fermions, as long as they are more than two sites from each other. Thus, one can view it as a repulsive potential for fermions, in addition to the hard core.

The supersymmetry is very useful for finding the $E = 0$ ground states of a theory [2]. All eigenvalues E of the Hamiltonian Eq. (1) satisfy $E \geq 0$, and the eigenstates form either singlet or doublet representations of the

of the value for close-packed dimers. For dimers on a two-leg ladder with $R + 1$ rungs, we found that the number N_{GS} of ground states grows quickly with R , and that not all ground states have the same dimer number, leading to (partial) cancellations in the Witten index W . For example, for $R = 7$, one finds five $E = 0$ ground states with $N_d = 5, 5, 5, 5, 6$, so $W = -3$. Our results, asymptotically precise for R large, are $N_{\text{GS}} \sim (1.395)^R$ and $W \sim (1.356)^R$. It will be interesting to see if supersymmetry can be of use in other quantum dimer models [5].

We now analyze the one-dimensional case in more detail. The Hamiltonian Eq. (4) is on an N -site chain:

$$H = \sum_{i=1}^N [P_{i-1}(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i)P_{i+2} + P_{i-1}P_{i+1}], \quad (6)$$

where $P_i \equiv 1 - c_i^\dagger c_i$ is the projector on a single site. We take periodic boundary conditions, so indices are defined mod N . The translation operator T commutes with both H and F , and its eigenvalues t satisfying $t^N = 1$ characterize the eigenstates of H . The Hamiltonian Eq. (6) resembles a lattice version of the Thirring model, a $(1 + 1)$ -dimensional field theory with a four-fermion interaction term. Below we will make the connection precise.

We originally obtained the space of states for this model by applying a systematic “finitization” procedure [6] to the chiral spectrum of a specific two-dimensional $\mathcal{N} = (2, 2)$ superconformal field theory (SCFT) with central charge $c = 1$. This construction follows two steps. In the first step, the full chiral Hilbert space of the SCFT is written in a “quasiparticle” basis, with the fundamental quasiparticles forming a supersymmetry doublet with charges $(1/3, -2/3)$. In the second step, the momenta of the individual quasiparticles are constrained to a maximum value in the order of N , corresponding to a discretization of the space direction of the SCFT with spacing of the order $1/N$. This then leads to a truncated or “finitized” partition sum $Q_N(q, w)$, which is closely related to the partition sum $Z_N(q, w) = \text{Tr}(q^{[N/(2\pi i)] \log T} w^{(N-3F)})$, which keeps track of the eigenvalues of the translation operator T and the fermion-number F of the lattice model Eq. (6). This entire construction, to be detailed elsewhere, respects the supersymmetry. This gives a rationale for the existence of supersymmetry generators on this space of states, and it provides an independent way of determining the quantum numbers t and f of the supersymmetric ground states. We remark that an analogous construction based on the spinon basis of the simplest $\text{SU}(2)$ -invariant CFT naturally leads to the space of states of the Heisenberg and Haldane-Shastry models for N spin-1/2 degrees of freedom. Clearly, many generalizations are possible.

We have discussed how the space of states in the model with Hamiltonian (6) arises from the finitization of the $c = 1$ conformal field theory. Moreover, the lattice model

and the SCFT have the same Witten index $W = 2$. We now give the results of a Bethe ansatz computation [7], which makes the correspondence precise.

An eigenstate of H with f fermions is of the form

$$\phi^{(f)} = \sum_{\{i_k\}} \varphi(i_1, i_2, \dots, i_f) c_{i_1}^\dagger c_{i_2}^\dagger \dots c_{i_f}^\dagger |0\rangle, \quad (7)$$

where we order $1 \leq i_1 < i_2 - 1 < i_3 - 2 \dots$. Bethe’s ansatz for these eigenstates is [8]

$$\varphi(i_1, i_2, \dots, i_f) = \sum_P A_P \mu_{P_1}^{i_1-1} \mu_{P_2}^{i_2-1} \dots \mu_{P_f}^{i_f-1}, \quad (8)$$

for some numbers $\{\mu_1, \dots, \mu_f\}$ and A_P ; the sum is over permutations P of the set $(1, 2, \dots, f)$. By construction, the translation operator T has eigenvalue $t = \prod_{k=1}^f \mu_k^{-1}$. For a generic model, this ansatz does not work, but in the Heisenberg and other integrable models, the miracle is that it does. Here, the ansatz gives eigenstates of energy

$$E = N - 2f + \sum_{k=1}^f \left[\mu_k + \frac{1}{\mu_k} \right], \quad (9)$$

when the μ_k obey the Bethe equations

$$t^{-1}(\mu_k)^{N-f} = \prod_{j=1}^f \frac{\mu_k \mu_j + 1 - \mu_k}{\mu_k \mu_j + 1 - \mu_j}, \quad (10)$$

for all $k = 1, \dots, f$. These are very similar to the Bethe equations for the antiferromagnetic XXZ spin chain at $\Delta = \pm 1/2$ [9]. The only difference in (10) is in the left-hand-side, which in the XXZ case reads $(\mu_k)^N \tau$, where $\tau \neq 1$ corresponds to twisted boundary conditions.

The supersymmetry doublets appear naturally within the Bethe ansatz: If (μ_1, \dots, μ_f) satisfies the Bethe equations, then the set $(1, \mu_1, \dots, \mu_f)$ also satisfies them. It is straightforward to check that the states associated with these two sets are related by $\phi^{(f+1)} = Q^+ \phi^{(f)}$, and that both sets have the same energy E .

The Bethe equations are f coupled polynomial equations of order N . They cannot be solved in closed form and, to make further progress, one usually needs to take N large. In our case, however, the supersymmetry allows us to derive more results from the Bethe ansatz for finite N . Precisely, we define w_k in terms of μ_k as $w_k = (\mu_k - q)/(q\mu_k - 1)$, where $q \equiv \exp(-i\pi/3)$. Then Baxter’s \mathcal{Q} function [9] $\mathcal{Q}(w) \equiv \prod_{i=1}^f (w - w_i)$ has zeros at $w = w_k$. Defining $\mathcal{R}(w) \equiv \mathcal{Q}(w)(1+w)^{N-f}$, we find that, for the w_k giving the ground state,

$$\mathcal{R}(q^{-2}w) = tq^{-N}\mathcal{R}(w) + t^{-1}q^N\mathcal{R}(q^2w). \quad (11)$$

We derive an explicit expression for $\mathcal{R}(w)$ in the sequel [7], but from (11) directly we can rederive f and t for the ground state(s). When $N = 3p$ with p an integer, there are nontrivial solutions to (11) only when $f = N/3$ and $t = (-1)^N \exp(\pm i\pi/3)$. For $N \neq 3p$, one has only a single solution with $f = \text{int}[(N + 1)/3]$, and $t = (-1)^{N-1}$.

We now discuss the field theory describing the continuum limit of (6). When N is large, one can rewrite the Bethe equations in terms of densities of roots [8], and then derive integral equations (known as thermodynamic Bethe ansatz equations) yielding the free energy. Our model has the same thermodynamic equations as the XXZ chain at $\Delta = 1/2$, so the two models coincide in the continuum limit. The continuum limit of the XXZ chain is described by the massless Thirring model [10], or, equivalently, a free massless boson Φ with action [11]

$$S = \frac{2g}{\pi} \int dx dt [(\partial_t \Phi)^2 - (\partial_x \Phi)^2].$$

The continuum limit of the $\Delta = 1/2$ model has $g = 2/3$; this is the simplest field theory with $\mathcal{N} = (2, 2)$ superconformal symmetry [11]. The $(2, 2)$ means that there are two left and two right-moving supersymmetries: In the continuum limit, the fermion decomposes into left- and right-moving components over the Fermi sea. The boson also can be decoupled into left and right pieces, so that $\Phi = \Phi_L + \Phi_R$, while its dual $\bar{\Phi} = g(\Phi_L - \Phi_R)$. The states of the field theory are given by the vertex operators $V_{m,n} = \exp(im\Phi + in\bar{\Phi})$, of conformal dimensions $h_{L,R} = (m \pm gn)^2 / (4g)$. The four components of the Dirac fermion in the Thirring model are $V_{\pm 1, \pm 1/2}$, while the supersymmetry generators are $Q_L^\pm = V_{\pm 1, \pm 3/2}$ and $Q_R^\pm = V_{\pm 1, \mp 3/2}$. In a finite size \mathcal{L} , the lowest-energy state is in the Neveu-Schwarz sector, where the Thirring fermion has antiperiodic boundary conditions. This state has $E_{NS} = -\pi/(6\mathcal{L})$. The lowest-energy states in the Ramond (periodic boundary conditions) sector are given by $|\pm\rangle_R = V_{0, \pm 1/2} |0\rangle_{NS}$; both have energy zero. States in this conformal field theory can be built up by operating with the “spinons” $V_{\pm 1/3, \pm 1/2}$.

Comparing this superconformal field theory with the lattice model on $N = 3p$ sites, we identify our two $E = 0$ ground states with the two Ramond vacua; all other states of the lattice model are in the Ramond sector as well. The $U(1)$ quantum number m corresponds to $f - N/3$ (the fermion number relative to the ground state). The spinons here have charge $\pm 1/3$, so it is natural to identify these with the fractionally charged excitations in the lattice model.

We close by noting several generalizations of our construction. Here the space of states is made up solely of fermions on which the supersymmetry acts nonlinearly, but in the sequel [7] we will study linear realizations as well. Other supersymmetric models arise by including more projectors in (3), or by including several species of fermions. Consider a two-species model, with fermions labeled by $+$ and $-$, and with the conditions that (i) a single site may not be occupied by two particles and (ii) same-type particles may not occupy nearest neighbor

sites. On a periodic chain with $4n$ sites, the Witten index of this model turns out to be $W = 3$, and we have strong indications that the continuum limit of this theory is the second model (at $c = 3/2$) of the series of $\mathcal{N} = 2$ minimal superconformal field theories. A typical ground-state pattern is

$$\dots \circ + \circ - \circ + \circ - \circ + \circ - \circ \dots$$

and one recognizes the possibility of domain walls of charge $\pm 1/2$ ($\circ + - \circ$) and neutral defects ($+ \circ +$). The $+/-$ pattern indicates an Ising substructure in the model, in accord with the fact that the $c = 3/2$ $\mathcal{N} = 2$ minimal model can be written in terms of a Majorana fermion and a free boson.

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