Gradient-Limited Surfaces: Formation of Geological Landscapes

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A simple scenario of the formation of geological landscapes is suggested, and the respective lattice model is derived. Numerical analysis shows that the arising non-Gaussian surfaces are characterized by the scale-dependent Hurst exponent varying from 0.7 to 1, in agreement with experimental data.

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Rough interfaces around us, such as Earth's surface [1,2], surfaces of deposited films [3], wetting fronts [4], cloud perimeters [5], fracture surfaces [6], etc., are common objects, the properties and formation of which have been studied for several decades. In some cases, significant advances in theoretical understanding have been achieved. In particular, this applies to the surface growth processes, the analysis of which has led to a wide variety of kinetic roughening models; cf. [7,8]. However, many processes leading to rough surfaces, e.g., the formation of fractures and Earth's landscapes, are less understood.

The formation of the Earth's surface is a complex process, affected by various phenomena, such as seismic and tectonic activity, erosion, sedimentation, etc. These phenomena, in their turn, can be of diverse nature. Thus, erosion can be caused by meandering rivers, oceanic and atmospheric influence, by the motion of ice, avalanches, and so on. Furthermore, the physical properties of the ground vary in a very wide range. Incorporating all this diversity into a concise embraceable mathematical model is a hopeless task. However, the scale-invariant properties of the geologic landscapes have been found to be surprisingly universal: in a reasonable approximation, they are typically self-affine, with the Hurst exponent ranging between $H \approx 0.7$ and 0.9 [1]. More recently, experimental evidence has been provided that the self-affine behavior is not perfect and the differential Hurst exponent h is a decreasing function of scale, $h \sim 0.8$ being characteristic to smaller scales; see Ref. [2] and references therein. So, it is natural to expect that there is a simple, universal, and robust mechanism leading to such surfaces. Here we show that such a mechanism can be provided by the competition of erosion and tectonic activity: the model of gradient-limited surfaces incorporates these two effects in their simplest form and leads to realistic landscapes.

Most models of geological landscapes are based on the evolution of river networks [9]; a more generic approach models the erosion on a slope via a stochastic equation (resulting in direction-dependent exponents H = 0.63 and 0.83) [2]. The evolution of rivers and erosion undoubtedly play an important role in the formation of landscapes, but are not able to increase the height of the mountains. The only attempt of including tectonic processes into a robust self-affine model of the Earth's sur-

face has been made by Mandelbrot [1]. He modeled roughening due to tectonic activity; the method can be outlined as follows. Inside a polygon (Earth's surface), a random point is coined. Through that point, a line of random direction is drawn. This "fault line" divides the polygon into two parts, one of which is elevated (with respect to the other) by a unit height. The procedure is repeated $N \rightarrow \infty$ times. The Brownian growth of height differences is eliminated by normalizing the surface height to \sqrt{N} . This results in a self-affine surface with H = 0.5. In order to address the discrepancy between the model and empirical values $H \approx 0.7$ –0.9, the model has been generalized by replacing the Heaviside profile of the "fault" by a profile with singularity (so that the height change across the fault $\Delta h = |x|^{\alpha} \operatorname{sgn} x, |\alpha| < \frac{1}{2}$).

The tectonic activity and formation of faults, as captured by the Mandelbrot model, certainly play an important role in the evolution of the Earth's surface. Meanwhile, singular fault profiles have no physical motivation, and there are no physical processes normalizing the surface height to the number of faults. Instead, the basic effect reducing the height differences is erosion. The excessively detailed erosion models, however, are not suited for revealing the most generic aspects of landscape roughening. Therefore, we opt for the simplest possible approach and assume that erosion effectively imposes an upper limit to the modulus of the gradient of the surface height. More specifically, we assume that, as soon as a slope becomes steeper than a threshold value, the excess of the height drop is spread over the neighboring regions. Such a smoothing of too steep slopes can be accomplished, for instance, by avalanches.

To begin with, let us define this model for a continuous medium. A random point P and direction τ define a fault line inside a polygon of diameter $L \gg 1$. This line divides the polygon into two parts, one of which (leftmost, with respect to the direction τ) is elevated by a unit height. The height drop is spread over the nearest neighboring regions in such a way that the modulus of the local gradient remains everywhere below a threshold value, i.e., $|\nabla \psi| < 1$. If the fault line goes through a region of a saturated slope, the avalanches can affect large areas. Then, the actual elevation (the change of surface slope) will take place far from the fault, at the edges of the

saturated slope. These edges will be referred to as the elevation lines. The procedure is repeated $N \rightarrow \infty$ times. This model has a lower cutoff scale, the height drop at a fault divided by the threshold gradient.

The model can also be formulated on a lattice. A natural basis for this is given by the six-vertex (6V, restricted solid-on-solid) model [3,10]. On the square lattice of the 6V model [Fig. 1(a)], all the edges are marked with arrows so that each vertex has an equal number of incoming and outgoing arrows. The arrows define an incompressible flow, the stream function of which is our surface; each arrow represents a unit jump in the surface height. The mean slope is defined by the mean density of unidirectional arrows, which is limited by the grid step; i.e., the slope steepness is constrained automatically.

Consider an oriented chain of arrows dividing the lattice into two parts [see Fig. 1(a)]. Swapping the direction of all the arrows of the chain is legitimate, because for all the affected vertices, the number of incoming arrows is conserved. It corresponds to the lowering of one part of the surface with respect to the other part by two units. Therefore, *directed chains of arrows* can play the role of elevation lines. Gradient-limited surfaces are obtained as follows. A random vertex of the lattice P and a random direction τ are coined, they define an *aim line*, the site of the "fault." That part of the surface, which is leftwards to the aim line, is to be elevated. The elevation is accomplished along such a directed chain of arrows, which follows as closely as possible the aim line. Similar to the continuous case, if the aim line goes through a region of counterdirected arrows (saturated slope), the elevation line is forced to go around those regions. There are different technical options of how to minimize the distance between the elevation line and the aim line. In



FIG. 1. The algorithm of finding the elevation line (bold) using the lattice of the 6V model (darker areas are lower). First, the origin P and direction τ of the aim line (dashed) are coined. The elevation line is such a directed chain of arrows, which follows the aim line as closely as possible. For the fastest algorithm, it is traced step by step, starting from the origin P, towards both ends of the aim line. At each step, there are two possibilities to continue the line, since there are two outgoing arrows from each vertex. The selection is based on two rules: (i) displacements opposite the axes of the quadrant [I–IV (b)] of the vector τ are not allowed (e.g., for quadrant I, leftwards motion is excluded); (ii) that option is to be selected, which leads closer to the aim line. The (signed) departure from the aim line is tracked as $s = x \sin \alpha - y \cos \alpha$, where x and y are the displacement coordinates, and α is the angle between the x axis and the aim line in (c). The elevation line is terminated as soon as it reaches the boundary of the polygon.

particular, it can be done locally and globally. However, the scaling properties of the resulting surfaces are insensitive with respect to the particular choice; for our main series of simulations, we used the fastest algorithm (described in Fig. 1). The gradient-limited surfaces are obtained at the long-time limit, when the number of elevations exceeds the relaxation time (which scales as the number of vertices in the polygon; see below), and the initial shape of the surface becomes irrelevant.

A simulation result is presented in Fig. 2. Observe the shape of the elevation line: in the region of a saturated slope, it has to depart far from the straight aim line. These regions are responsible for the long-range correlation of the surface height increments. Qualitatively, the saturated slopes are caused by accumulated excess of the "faults" of a certain direction. If there were no avalanches, that excess would fluctuate as the square root of the number of "faults," tending to infinity. Therefore, the presence of large saturated slopes should not be surprising. As it will be shown below, in a one-dimensional (1D) case, the accumulation phenomenon gives rise to H = 1 (i.e., typically, a saturated slope occupies the whole polygon). In 2D geometry, the accumulation effect is weaker and leads to a complex (inaccessible with standard analytic methods) scaling behavior.

The 1D version of the model is most conveniently formulated as a spin exchange problem, and admits analytical solution. This analytic approach helps us to understand the features of the 2D model. Suppose there is a sequence of spins, $\varphi_i = \pm 1$. It is convenient to consider infinite periodic sequence, $\varphi_{i+N} = \varphi_i$, where $i \in Z$ and



FIG. 2. A gradient-limited surface, polygon size $L_{\text{max}} = 2049$. Darker areas correspond to lower regions of the surface, and black and white lines depict equidistant level lines. Black line surrounded by white is an elevation line.

N is the period. The spins φ_i can be interpreted as the increments of a self-affine curve $\psi_j = \sum_{i=0}^{J} \varphi_i$. A random point k and a random spin increment $\nu = \pm 2$ are coined. The increment is to be added to the kth spin, or, if it is not possible (resulting in $\varphi_k = \pm 3$), to the nearest suitable spin (i.e., to $\varphi_l = -\nu \pm 1$ with a minimal value of |l-k|). If there is no suitable spin at all, the next pair of k and ν are coined. The procedure is repeated ad infinitum. Let us denote the relative number of positive spins by ξ . Then, the height drop of the above defined self-affine curve ψ_i at distance N is $N|2\xi - 1|$. The quantity ξ performs Brownian fluctuations, because at each time step it is randomly incremented by $\pm N^{-1}$. At the limit $N \rightarrow \infty$, the probability density function $n(\xi, t)$ evolves according to the diffusion equation, $n_t = Dn_{\xi\xi}$ with D = $(2N^2)^{-1}$ and no flux at the boundaries, $n_{\xi}(0, t) =$ $n_{\xi}(1, t) = 0$. The stationary solution $n(\xi, t) \equiv 1$ allows us to calculate the delta variance $\langle (\psi_{i+N} - \psi_i)^2 \rangle =$ $N^2 \int_0^1 (2\xi - 1)^2 d\xi = N^2/3$, which corresponds to H = 1. The relaxation time of the spin exchange problem can be found as the diffusion time, $\tau \approx N^2$ (time is measured in the number of spin exchanges). The relaxation time of the 2D model is estimated in the same way, because the height difference between left and right edges of the polygon performs also nearly Brownian fluctuations (however, the Brownian behavior breaks up for nearly saturated slopes: those elevation lines, which would make the slope steeper, tend to incline outside the polygon).

For 2D geometry, the simulations indicate that the gradient-limited surfaces are not strictly speaking selfaffine. However, the data collapse is achieved by introducing the differential Hurst exponent,

$$h(\lambda) = \frac{1}{2} d \log \langle a_L^2 \rangle / d \log L, \qquad \lambda = \log L / \log L_{\text{max}}.$$
 (1)

Here a_L is the height of the surface at the distance L from the center of the polygon of size L_{max} ; angular braces denote averaging over different realizations of the surface. In Fig. 3, the differential Hurst exponent is approximated by $\tilde{h}(\lambda, L_{\text{max}}) = \log(\langle a_i^2 \rangle / \langle a_{i+1}^2 \rangle) [\log(L_i/L_{i+1})]^{-1}$, where *i* and *i* + 1 are neighboring data points and $\lambda =$ $\log(L_iL_{i+1})/\log L_{\max}^2$. At the limit $L_{\max} \to \infty$, the curves converge to the asymptotic function $h(\lambda)$. Note that at the extreme right-hand side of the plot, the $\tilde{h}(\lambda, L_{max})$ curves show a rapid falloff, and the convergence is not as good as elsewhere; this is explained by finite-size effects and by the fact that for $\lambda \approx 1$, the finite differences fail providing an acceptable approximation for the derivative in Eq. (1). As a consequence, the values $\tilde{h} \leq 0.65$ are not reliable. For large scales with $\lambda \ge 0.9$ (which are most interesting in the context of the Earth's surface), the $h(\lambda)$ law can be found by extrapolating the asymptotic curve [dashed line in Fig. 3(a)]. The conclusion $h \approx 0.7-0.9$ for $0.5 < \lambda < 1$ is in good agreement with the geological observations [1,2] (intriguingly, similar values are recorded for the fracture surfaces [6,11,12]). Experimental



FIG. 3. Gradient-limited surfaces: numerical results. The numbers 65–1025 indicate the edge length L_{max} of the polygon. The finite difference approximation of the differential roughness exponent \tilde{h} is plotted versus the logarithmic relative scale λ . There is no strict self-affinity of the surface. However, at the limit $L_{\text{max}} \rightarrow \infty$, there is an asymptotic dependence $\tilde{h}(\lambda, L_{\text{max}}) \rightarrow h(\lambda)$ (dotted line). The asymptotic convergence is clearest in (b), where the integral exponent \tilde{H} is plotted against the polygon size. The simulations have been carried out with an assembly-optimized code on the cluster of ten 1-GHz Athlon workstations during two months (covering 2×10^5 decorrelation times for $L_{\text{max}} = 1025$).

data also confirm that \hat{h} increases towards small scales [2]; see also Fig. 4(a). The scale dependence of h is a result of interaction between tectonic processes and erosion: incorporating only the first [1] or the second process [2] leads to scale-invariant values of h.

Finally, the scaling of the edge-to-edge height drop of the gradient-limited surfaces is given by the integral Hurst exponent $H = \int_0^1 h d\lambda$: $\langle a(L_{\text{max}})^2 \rangle \propto L_{\text{max}}^{2H}$. The numerical result $H = 0.91 \pm 0.01$ [see Fig. 3(b)] shows the



FIG. 4. The data of real landscapes [13] support the two most important features of the gradient limited surfaces: (a) differential Hurst exponent increases towards small scales tending to h = 1, and (b) small contour lines occupy less space than large ones implying that the size-distribution exponent k > 0. The same trends have been observed for all the analyzed mountain regions (Himalayas, Pamirs, Caucasus, etc.). Note that the observation k > 0 provides a direct evidence for the qualitatively significant non-Gaussianity of these landscapes. The very small roughness $h \sim 0.4$ at large scales can be explained by the domination of other mechanisms (possibly reducible to the Kardar-Parisi-Zhang model) at these scales [2].



FIG. 5. Differential scaling exponents of the contour loops of gradient-limited surfaces. Finite difference approximation of the fractal dimension of a single loop \tilde{d} (a) and of the contour loops size-distribution exponent \tilde{k} (b) are plotted versus the scale λ . The numbers 65–1025 indicate the polygon size L_{max} . The dotted line depicts the asymptotic dependence $\tilde{d}(\lambda, L_{\text{max}}) \rightarrow d(\lambda)$. The dashed line is calculated using the dependence $h(\lambda)$ [see Fig. 3(a)], and the fractal dimension for Gaussian self-affine surfaces. Positive values of \tilde{k} indicate that there is an anomalously small number of small contour loops (as compared with the Gaussian surfaces).

existence of a nontrivial asymptotic curve $h(\lambda)$ (since $\int_0^1 h d\lambda \approx 0.91$), making the model clearly distinct from all the known universality classes, and has been used to test the asymptotic (dotted) curve in Fig. 3(a).

The observed generalized scale invariance with critical exponents depending on scale is not unique. For instance, similar behavior has been observed for certain forest fire models [14]. In our case, the Hurst exponent increasing towards small scales is caused by the presence of large areas of saturated slope. Indeed, consider a random pair of points. If the distance *L* between them is small, the points are likely to reside inside a single region of saturated slope. Hence, their average height difference scales almost as *L*, implying $h \approx 1$. On the other hand, larger saturated slopes are rarer than the smaller ones. Therefore, for a more distant pair of points, falling inside a single saturated slope is a rare event, and the conclusion $h \approx 1$ is no more valid.

For Gaussian self-affine surfaces, all the scaling exponents of statistical topography are functions of the Hurst exponent *H*. However, the gradient-limited surfaces are not Gaussian, as evidenced by the presence of large saturated slopes. Therefore, the exponent $h(\lambda)$ alone does not provide a complete description of the surface. First we consider the differential fractal dimension of the contour loops ("coastlines"), $d = d \log \langle l_L \rangle / d \log L$ (*l* is the length, and *L* is the diameter of a loop). The numerical results are given in Fig. 5(a). For Gaussian surfaces, the fractal dimension of contour loops $D(H) \approx 1.5 - 0.5H$ [8,15]. The dotted line [Fig. 5(a)] is the asymptotic ($L_{\text{max}} \rightarrow \infty$) dependence $d(\lambda)$, and the dashed line is the curve, calculated on the basis of the functions $h(\lambda)$ (Fig. 3) and D(H). Evidently, $D[h(\lambda)] \neq d(\lambda)$.

An even more pronounced mismatch between the Gaussian surfaces and the gradient-limited surfaces is observed for size distribution of the contour loops. Let p(L) denote the probability that a randomly chosen point belongs to such a contour loop, the diameter of which is between L and 2L. Then, for Gaussian surfaces we would expect that $p(L) \propto L^k$, where k = D(H) - (2 - H) < 0[8,16]. For gradient-limited surfaces, a convergence of the $\tilde{k}(\lambda, L_{\text{max}})$ curves to the asymptotic dependence $k(\lambda) > 0$ is clearly observed up to $\lambda \approx 0.85$ [see Fig. 5(b)]; above that scale, convergence is slow due to finite-size effects. The inequality k > 0 means that, as compared with the Gaussian surfaces, there is a significantly smaller number of small contour loops, and is explained by the fact that the saturated slopes can be embraced by large contour loops, but leave almost no room for small ones. Exactly such a non-Gaussian behavior is observed for real landscapes; see Fig. 4(b).

In conclusion, the new model of gradient-limited surfaces leads to non-Gaussian surfaces of a scale-dependent differential Hurst exponent. The latter varies from $h \approx 1$ for small scales, up to $h \approx 0.7$ for large scales. This is in reasonable agreement with the experimentally observed roughness of real geological landscapes.

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