Exact Self-Similar Solutions of the Generalized Nonlinear Schrödinger Equation with Distributed Coefficients

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A broad class of exact self-similar solutions to the nonlinear Schrödinger equation (NLSE) with distributed dispersion, nonlinearity, and gain or loss has been found. Appropriate solitary wave solutions applying to propagation in optical fibers and optical fiber amplifiers with these distributed parameters have also been studied. These solutions exist for physically realistic dispersion and nonlinearity profiles in a fiber with anomalous group velocity dispersion. They correspond either to compressing or spreading solitary pulses which maintain a linear chirp or to chirped oscillatory solutions. The stability of these solutions has been confirmed by numerical simulations of the NLSE.

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Studies of self-similar solutions of the relevant nonlinear differential equations have been of great value in understanding widely different nonlinear physical phenomena [1]. Although self-similar solutions have been extensively studied in fields such as hydrodynamics and quantum field theory, their application in optics has not been widespread. Some important results have, however, been obtained, with previous theoretical studies considering self-similar behavior in radial pattern formation [2], stimulated Raman scattering [3], the evolution of self-written waveguides [4], the formation of Cantor set fractals in soliton systems [5], the nonlinear propagation of pulses with parabolic intensity profiles in optical fibers with normal dispersion [6], and nonlinear compression of chirped solitary waves [7,8].

In this Letter we present the discovery of a broad class of exact self-similar solutions to the nonlinear Schrödinger equation with gain or loss (the generalized NLSE) where all parameters are functions of the distance variable. This class also encloses the set of solitary wave solutions which describes, for example, such physically important applications as the amplification and compression of pulses in optical fiber amplifiers [9]. These linearly chirped solitary wave solutions apply in the anomalous dispersion regime and may be contrasted with the asymptotic solutions appropriate in the normal dispersion regime [10,11]. The importance of the results reported here is twofold: first, the approach leads to a broad class of exact solutions to the nonlinear differential equation in a systematic way. Some of these solutions have been obtained serendipitously in the past, but we emphasize the importance of the use of self-similarity techniques which are broadly applicable for finding solutions to a range of nonlinear partial differential equations, having applications in a variety of other physical situations. These equations are not integrable by the inverse scattering method, and, therefore, they do not have soliton solutions [9]; however, they do have solitary wave solutions which have often been called solitons. The second and more specific significance of these results lies in their potential application to the design of fiber optic amplifiers, optical pulse compressors, and solitary wave based communications links.

The nonlinear Schrödinger equation with gain in the form used in nonlinear fiber optics is given by

$$i\psi_z = \frac{\beta(z)}{2}\psi_{\tau\tau} - \gamma(z)|\psi|^2\psi + i\frac{g(z)}{2}\psi, \tag{1}$$

where we suppose that all parameters β, γ, and g are functions of the propagation distance z. This equation describes the amplification or attenuation [when $g(z)$ is negative] of pulses propagating nonlinearly in a single mode optical fiber where $\psi(z,\tau)$ is the complex envelope of the electrical field in a comoving frame, τ is the retarded time, $\beta(z)$ is the group velocity dispersion (GVD) parameter, $\gamma(z)$ is the nonlinearity parameter, and $g(z)$ is the distributed gain function. In the absence of gain this equation has the well known soliton solutions when the dispersion and nonlinearity are constant. Furthermore, it is known that in the presence of constant loss or gain, the familiar (unchirped) soliton pulse shape can be maintained by propagating through a fiber with exponentially varying nonlinearity. This follows using Eq. (1) without gain but with constant dispersion and nonlinearity using the transformation $\psi' = \exp(\frac{g}{2}z)\psi$. We are concerned here, however, with solutions characterized by a linear chirp.

The complex function $\psi(z,\tau)$ can be written as

$$\psi(z,\tau) = U(z,\tau)\exp[i\Phi(z,\tau)], \tag{2}$$

where U and Φ are real functions of z and τ. We look for self-similar solutions of the NLSE assuming that the phase has a quadratic form:

$$\Phi(z,\tau) = a(z) + c(z)(\tau - \tau_c)^2. \tag{3}$$

This ansatz follows from the analysis of the existence of self-similar solutions to the equation and the value of this approach has been confirmed for any gain function in the case when $\beta\gamma > 0$ [11]. Equations (1)–(3) yield a self-similar form of the amplitude [12]:

$$U(z,\tau) = \frac{1}{\sqrt{1 - c_0 D(z)}} F\left(\frac{\tau - \tau_c}{1 - c_0 D(z)}\right) \exp\left(\frac{1}{2} G(z)\right), \quad (4)$$

where τ_c is the center of the pulse, and the functions $a(z)$, $c(z)$, $D(z)$, and $G(z)$ in the solutions given by Eqs. (3) and (4) are

$$a(z) = a_0 - \frac{\lambda}{2} \int_0^z \frac{\beta(z')dz'}{(1 - c_0 D(z'))^2}, \quad (5)$$

$$c(z) = \frac{c_0}{1 - c_0 D(z)}, \quad (6)$$

$$D(z) = 2\int_0^z \beta(z')dz', \qquad G(z) = \int_0^z g(z')dz', \quad (7)$$

where a_0, λ, and c_0 are the integration constants.

The necessary and sufficient condition for the existence of such self-similar solutions is given by the following relationship between the gain profile and the parameters $\rho(z) \equiv \beta(z)/\gamma(z)$ and $\beta(z)$ [12]:

$$g(z) = \frac{1}{\rho(z)} \frac{d}{dz} \rho(z) + \frac{2c_0 \beta(z)}{1 - c_0 D(z)}, \quad (8)$$

where $c_0 \neq 0$. The function $F(T)$ which determines the amplitude $U(z,\tau)$ in Eq. (4) can be found by solving the nonlinear differential equation

$$\frac{d^2F}{dT^2} - \lambda F + 2\kappa F^3 = 0, \quad (9)$$

where the scaling variable T and the coefficient κ are

$$T = \frac{\tau - \tau_c}{1 - c_0 D(z)}, \qquad \kappa = -\frac{1}{\rho(0)} = -\frac{\gamma(0)}{\beta(0)}. \quad (10)$$

Integrating Eq. (9) for the case $\beta(z)\gamma(z) < 0$ and using Eq. (4) we find the amplitude of the solitary wave solution:

$$U(z,\tau) = \frac{\sqrt{|\rho(z)|}}{\tau_0(1 - c_0 D(z))} \operatorname{sech}\left(\frac{\tau - \tau_c}{\tau_0(1 - c_0 D(z))}\right). \quad (11)$$

The integration constant λ for this case is $\lambda = \tau_0^{-2}$, where τ_0 is the initial pulse width. Another so-called kink solution follows from Eq. (9) with the condition $\beta(z)\gamma(z) > 0$ and yields the amplitude in the form

$$U(z,\tau) = \frac{\sqrt{\rho(z)}}{\tau_0(1 - c_0 D(z))} \tanh\left(\frac{\tau - \tau_c}{\tau_0(1 - c_0 D(z))}\right). \quad (12)$$

For this case $\lambda = -2/\tau_0^2$. The homogeneous solution (independent of τ) also follows from Eq. (9) for either sign of the product $\beta(z)\gamma(z)$ and is

$$U(z,\tau) = \frac{\Lambda\sqrt{|\rho(z)|}}{1 - c_0 D(z)}, \quad (13)$$

where Λ is an arbitrary real parameter and $\lambda = -2\Lambda^2 \operatorname{sgn}\rho(0)$. We note that Eq. (9) also has six bounded periodic solutions which are proportional to Jacobian elliptic functions: $\mathrm{sn}(\xi,k)$ and $\mathrm{cd}(\xi,k)$ for $\beta\gamma > 0$, $\mathrm{cn}(\xi,k)$, $\mathrm{dn}(\xi,k)$, $\mathrm{sd}(\xi,k)$, and $\mathrm{nd}(\xi,k)$ at $\beta\gamma < 0$, where $\xi = T/\tau_0$, and k is an arbitrary parameter in the interval $0 < k < 1$. Hence by Eq. (4) these solutions yield six bounded periodic solutions for the amplitude $U(z,\tau)$ with appropriate constants $\lambda \sim \tau_0^{-2}$. In the case $\beta(z) =$ const the traveling solutions can also be found by a Galilean transformation. These oscillatory solutions correspond to the transmission of amplitude modulated light which experiences a simultaneous amplification (or attenuation) and frequency shift under the influence of nonlinearity, dispersion, and gain (or attenuation). These solutions could provide a new technique for generating amplitude modulated light at THz frequencies.

We note that in the case when the dispersion and nonlinearity are constants, Eq. (8) yields the distributed gain $g(z) = 2c_0\beta(1 - 2c_0\beta z)^{-1}$. For this particular case the solitary wave solution has been reported in [7] which by transformation also yields the solution for propagation with an exponentially distributed dispersion parameter and constant gain. The numerical analysis of pulse compression in an exponentially decreasing dispersion profile has also been investigated in [13], while soliton pulse compression with adiabatic gain profiles has been demonstrated experimentally by using fiber Raman amplifiers [14]. The more general solutions reported here are particularly useful in the design of amplifying or attenuating pulse compressors for chirped solitary waves.

Returning now to the solitary wave solution of the generalized NLSE (with gain term) let us consider the compression problem of the laser pulse in a dispersion decreasing optical fiber. For definiteness we assume that the GVD and the nonlinearity parameter are distributed according to

$$\beta(z) = \beta_0 \exp(-\sigma z), \qquad \gamma(z) = \gamma_0 \exp(\alpha z), \quad (14)$$

where $\beta_0 < 0$, $\gamma_0 > 0$, and $\sigma \neq 0$ ($\sigma > 0$ for dispersion decreasing fibers). As follows from Eq. (8) the gain function for self-similar solutions in this case is

$$g(z) = -\alpha - \frac{\sigma(\nu - 1)}{\nu - 1 + \exp(-\sigma z)}, \quad (15)$$

where we have introduced the parameter $\nu \equiv \sigma(2c_0\beta_0)^{-1}$. Hence the phase functions $a(z)$ and $c(z)$ given by Eqs. (5) and (6) at $\lambda = \tau_0^{-2}$ can be written

$$a(z) = a_0 - \frac{\beta_0 \nu^2}{2\sigma\tau_0^2}\left(\frac{1}{\nu - 1 + \exp(-\sigma z)} - \frac{1}{\nu}\right), \quad (16)$$

$$c(z) = \frac{\sigma}{2\beta_0}\left(\frac{1}{\nu - 1 + \exp(-\sigma z)}\right), \quad (17)$$

where σ and ν are arbitrary (nonzero) parameters. The amplitude of the solitary wave solution given by Eq. (11) in this case is

$$U(z, \tau) = A(z)\mathrm{sech}\left(\frac{\tau - \tau_c}{W(z)}\right), \quad (18)$$

where the scaling of the peak amplitude $A(z)$ and the pulse width $W(z)$ are given by

$$A(z) = \frac{\nu}{\tau_0}\sqrt{\frac{|\beta_0|}{|\gamma_0|}}\left(\frac{\exp[-\frac{1}{2}(\alpha + \sigma)z]}{\nu - 1 + \exp(-\sigma z)}\right), \quad (19)$$

$$W(z) = \frac{\tau_0}{\nu}[\nu - 1 + \exp(-\sigma z)]. \quad (20)$$

Let us consider the most typical physical situation when the loss in an optical fiber is a constant. According to Eq. (15) this takes place at $\nu = 1$ ($\sigma \neq 0$) and $\alpha > 0$; hence the gain $g(z) = -\alpha$ is negative. It is remarkable that [as follows from Eqs. (18) and (20)] the width of the sech pulse at $\nu = 1$ and $\sigma > 0$ tends to zero when z goes to infinity. This means that the case $\beta_0 < 0$, $\sigma > 0$, $\gamma_0 > 0$, $\alpha > 0$, and $\nu = 1$ provides the optimal situation for pulse compression. In fact Eq. (18) shows that $|\psi(z, \tau)|^2 \to E_0 \exp(-\alpha z)\delta(\tau - \tau_c)$ as $z \to \infty$, where $E_0 = 2|\beta_0|(\tau_0|\gamma_0|)^{-1}$ is the input energy of the pulse and $\delta(\tau - \tau_0)$ is the singular Dirac function. Moreover, there are three cases which follow from Eq. (19) (when $\nu = 1$, $\sigma > 0$, $\alpha > 0$) for the peak amplitude $A(z)$: (i) $\sigma > \alpha$, $A(z)$ grows for increasing z, (ii) $\sigma < \alpha$, $A(z)$ decreases for increasing z, and (iii) $\sigma = \alpha$, $A(z) = \mathrm{const}$.

A key consequence of this exact solution for constant loss is that the pulse can be compressed to any required degree as $z \to \infty$ while maintaining its sech shape and linear chirp in the presence of an exponentially distributed dispersion and nonlinearity parameter [see Eqs. (14)]. This analysis supposes that we neglect the higher order terms in the generalized NLSE, which will ultimately limit the amount of compression that can be obtained.

The solution given by Eqs. (16)–(20) at $\alpha < 0$ can also be applied to the problem of pulse compression in optical fiber amplifiers. In the case $\beta_0 < 0$, $\sigma > 0$, $\gamma_0 > 0$, $\alpha < 0$, and $\nu = 1$ the gain $g(z) = -\alpha$ is positive and hence the energy of the pulse grows as $E(z) = E_0 \exp(-\alpha z)$, but the width of sech pulse tends to zero as $W(z) = \tau_0 \exp(-\sigma z)$ when $z \to \infty$. From Eq. (14) it follows that in this case we should use a decreasing nonlinearity parameter together with a decreasing GVD parameter in a fiber amplifier to compress (theoretically) the sech pulse to any required width. This is to be contrasted with previous attempts to simulate the nonlinear amplification of pulses in the anomalous dispersion regime using constant parameters, which have shown that the pulse tends to break up into a series of pulses due to the combined effects of self-phase modulation and dispersion. Allowing the amplifier to have a distributed gain profile ($\nu \neq 0$) provides other design possibilities for an amplifying pulse compressor. From this solution it also follows that for $\sigma < 0$ (corresponding to an increasing dispersion parameter $|\beta(z)|$), the chirped sech pulse will spread for both amplification and attenuation. It is important to realize that the general class of solutions reported here includes those applying to some physically realistic constraints such as assuming that any two of the three distributed parameters, for example the gain as in (7), be held constant. More creative possibilities involve allowing two of these parameters to be appropriately engineered while one is held constant.

We point out that the case $\sigma > 0$, $\alpha = 0$, $\nu > 1$, and $g(z) = -\sigma(\nu - 1)/[\nu - 1 + \exp(-\sigma z)]$ has application to long-haul chirped soliton links where fiber losses are compensated periodically by an amplification system. This long-haul link is based on a distributed dispersion-loss-managed chirped soliton propagation regime and is an alternative to loss-managed soliton systems [15]. The main advantage of such systems is the absence of soliton radiation due to the fact that solitary waves propagating in this regime are an exact solution of Eq. (1) and hence generate no radiative noise.

The self-similar solitary wave solutions derived here correspond to hyperbolic secant pulses with a linear chirp which propagate without change in shape, only a scaling of their width and amplitude. These self-similar pulses maintain their linear chirp, although this chirp increases when $\sigma > 0$ ($\nu = 1$) and decreases when $\sigma < 0$. These features are shown in Fig. 1 where the analytical solution is compared with the results of numerical propagation of a hyperbolic secant pulse with the correct initial amplitude and chirp through the fiber with the distributed parameters given in Eqs. (14) and (15). In these simulations, the input pulse has an initial width of 7 ps and energy of 0.5 pJ, and the amplifier parameters are $\beta_0 = -0.01\ \mathrm{ps^2\,m^{-1}}$, $\gamma_0 = 0.01\ \mathrm{W^{-1}\,m^{-1}}$. The pulse profile is plotted at a propagation length of 10 m for (a) $\sigma = 0.1$, $\alpha = -0.1$ (an attenuated solution) and (b) $\sigma = 0.1$, $\alpha = 0.1$ (distributed amplification). Clearly the numerical simulations are in excellent agreement with the analytical predictions and in other simulations the pulse has been shown to compress by orders of magnitude in a few meters of fiber. It must be emphasized, however, that before these levels of compression are reached in a real amplifier or attenuator, the pulse evolution would be affected by higher order terms in the NLSE which are neglected in deriving these solutions.

Let us consider now the case $\beta = \beta_0 \exp(g_0 z)$, $\gamma(z) = \gamma_0$, $g(z) = g_0$ ($\beta_0\gamma_0 < 0$). For these parameters the necessary condition given by Eq. (8) is not satisfied for all z; however, this condition is complied within the interval $z \in (0, \varepsilon)$ at the limit when $\varepsilon \to 0$, $g_0 \to \infty$, and

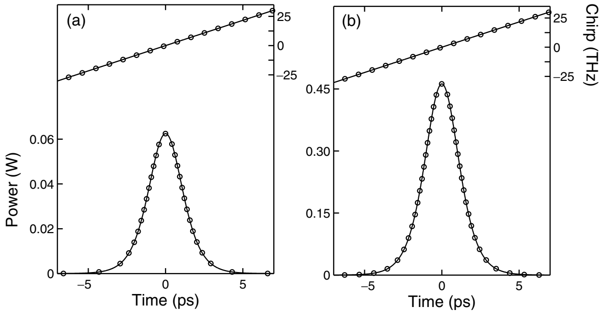

FIG. 1. Comparison of analytical solution (circles) with numerical simulation (solid line) for a propagating solitary pulse in an amplifier with constant loss (a) or gain (b) and distributed dispersion and nonlinearity.

$\lim_{\varepsilon\to 0} g_0\varepsilon = G$. Hence using Eqs. (5), (6), and (11) we find an exact solution of this problem (when $\varepsilon \to 0$) for output pulse $\psi_{\rm out}(\tau)$:

$$\psi_{\rm out}(\tau) = \exp\left(\frac{G}{2}\right)\psi_{\rm in}(\tau), \qquad \psi_{\rm in}(\tau) = \frac{\sqrt{|\rho_0|}}{\tau_0}\,{\rm sech}\left(\frac{\tau-\tau_c}{\tau_0}\right)\exp[i\Phi_0(\tau)], \tag{21}$$

where $\psi_{\rm in}(\tau)$ is input pulse at $z = 0$, $\rho_0 = \beta_0/\gamma_0$, and $\Phi_0(\tau) = a_0 + c_0(\tau - \tau_c)^2$. This result can be applied to an optical amplifier when the length ε is small and the gain $g_0 = G/\varepsilon$ is large but G is limited (not large); then according to Eq. (21) the output pulse is also a chirped soliton with the same width τ_0 and phase $\Phi_0(\tau)$; however, it has a new amplitude; i.e., the amplifier acts linearly on the pulse. Note that this result is also correct when $g_0 < 0$ and $G < 0$. Thus we have found an exact solution of Eq. (1) with singular distributed parameters

$$\beta(z) = \beta_0 \exp[G\delta_\varepsilon(z - \varepsilon/2)z], \qquad \gamma(z) = \gamma_0, \qquad g(z) = G\delta_\varepsilon(z - \varepsilon/2), \tag{22}$$

where $\delta_\varepsilon(z)$ is delta sequence defined as $\delta_\varepsilon(z) = \varepsilon^{-1}$ at $z \in (-\varepsilon/2, \varepsilon/2)$ and $\delta_\varepsilon(z) = 0$ at $z \notin (-\varepsilon/2, \varepsilon/2)$. We note that the usual functional definition of a delta function for this nonlinear problem is inapplicable since the product of such distributions is undefinable [16]; however, for our delta sequence any exponentials are well determined: $\delta_\varepsilon(z)^n = \varepsilon^{-(n-1)}\delta_\varepsilon(z)$.

We have also performed numerical simulations to determine the stability of the evolution of these solitary pulses in the presence of perturbations and nonideal parameter profiles. The evolution is more sensitive to the initial chirp than the peak amplitude, but in both cases the addition of small amounts of random noise to the input pulse amplitude and phase did not significantly affect the evolution. Indeed in both cases the pulses evolved towards the ideal form, indicating the stability of the solution.

The exact effects of the higher order terms in the NLSE, which will need to be considered as the pulse compresses towards zero width, and the intensity and pulse width at which these terms become important will be the subject of future numerical investigations. Preliminary investigations have shown, however, that the presence of the linear chirp minimizes the effects of stimulated Raman scattering which causes the self-frequency shift in the case of unchirped soliton pulses.

The results given here have been found by the systematic method [12] and represent the most general class of self-similar solutions to the NLSE with distributed coefficients, which can be readily applied to pulse propagation in nonlinear optical fiber amplifiers, optical fiber compressors, and long-haul links with distributed dispersion-loss-managed chirped solitons. Since linearly chirped pulses can readily be generated and compensated, these newly found propagation regimes may find significant applications.

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