Landau Theory of Shape Phase Transitions in the Cranked Interacting Boson Model

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Landau theory of phase transitions is applied to quadrupole shapes of rotating atomic nuclei within the interacting boson model (IBM) with cranking. It is shown that the coherent-state method must be generalized to allow for non-Hermitian quadrupole tensors of the coherent-state coefficients, which results in important modifications of the cranking shape-phase diagram compared to previous non-IBM studies of rotating nuclei. The parameter space has two surfaces of the first-order phase transitions and a curve of the second-order phase transition at their intersection. The phase structure of the cranked IBM closely resembles systems with competing superconducting and normal phases.

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It is known that the Landau theory of phase transitions [1,2] provides a general framework for the description of a crossover between various types of nuclear shapes. It has been successfully applied to quadrupole deformations of hot rotating nuclei in a series of papers by Alhassid, Levit, and Zingman [3–5]. Recent analysis [6] of the interacting boson model (IBM) [7] within the Landau theory showed that the phase diagram of nuclear deformations exhibits an isolated point of the second-order phase transition, a "triple point" located at the intersection of two first-order phase-transitional lines, namely, the borders between spherical and deformed, and between prolate and oblate axially symmetric shapes. In this Letter, we investigate an extension of the latter result to rotating nuclei by analyzing the phase-transitional behavior of the IBM with cranking.

The interacting boson model [7] describes low-energy collective excitations of even nuclei in terms of bosons with angular momenta 0 and 2, so-called s and d bosons. The bosonic Hamiltonian is assumed to have a general form with one- and two-body terms and must be invariant under some fundamental symmetries. The algebraic formulation of the IBM allows one to extract several alternative dynamical symmetries and find analytical solutions in case the Hamiltonian has any of these symmetries. The geometric content of the model follows from the group-theoretical procedure involving coherent states [8–10]. In particular, the U(5) dynamical-symmetry limit corresponds to spherical nuclear shapes, the SU(3) and $\overline{\mathrm{SU}(3)}$ limits to axially symmetric deformed shapes, prolate and oblate, respectively, while both the SO(6) and $\overline{\mathrm{SO}(6)}$ limits describe deformed nuclei allowing the onset of triaxiality. The geometric analysis assigns to the IBM similar meaning as to the older geometric collective model, but, in the same time, the IBM bosons have an apparent microscopic origin in nucleonic Cooper pairs, so that their fixed total number properly expresses some of the finite-size quantum effects in realistic nuclei.

Various aspects of the IBM with cranking were discussed in Refs. [11–14] and recently also in Ref. [15]. It seems that the shape-phase analysis of the IBM in a rotating frame needs just to reiterate the procedure from Refs. [3–5]. However, as shown in Refs. [14,15], to extract geometric properties of the IBM with cranking requires a specific approach, which leads to some important deviations from previous results. The modified phase structure of the cranked IBM will be discussed in this Letter from the viewpoint of the Landau theory.

The general Landau theory of shape-phase transitions in hot rotating nuclei, as elaborated in Refs. [3–5], starts by expressing the nuclear free energy F through rotational invariants built from the quadrupole shape variables $\alpha^{(2)}$ (that form a second-rank tensor) and the cranking frequency $\vec{\omega}$ (which is a first-rank tensor, $\omega^{(1)}$). In this way, F is written as a series of terms with increasing powers of $\alpha^{(2)}$ and $\omega^{(1)}$, where only the weight coefficients at each term are model dependent (they carry the dependence on the remaining scalar parameters, such as temperature T). It is essential that the invariants of the first-order in ω vanish, while the terms proportional to ω^2 define the tensor of inertia (which is in this way given by $L = 2$ coupled products of $\alpha^{(2)}$).

This procedure yields the following expression [4,5] for the free energy:

$$F(T,\beta,\gamma) = F_0(T) + A(T)\beta^2 - B(T)\beta^3\cos 3\gamma + C(T)\beta^4 - I_{zz}(T,\beta,\gamma)\omega^2/2 + \dots, \quad (1)$$

where dots stand for higher-order terms and I_{zz} is the largest diagonal component of the moment of inertia in the principal frame, $I_{zz}(T,\beta,\gamma) = I_0(T) - 2R(T)\beta\cos\gamma + 2I_1(T)\beta^2 + 2D(T)\beta^2\sin^2\gamma + \dots$. Hill-Wheeler quadrupole deformation parameters β and γ are given by the quadrupole variables in the principal frame, $\alpha_0 = \beta\cos\gamma$, $\alpha_{\pm 1} = 0$, $\sqrt{2}\,\alpha_{\pm 2} = \beta\sin\gamma$, and become the order parameters that characterize individual shape phases: spherical for $\beta = 0$, deformed axially symmetric for $\beta \neq 0, \gamma = 0$ (prolate and oblate shapes are in this Letter distinguished by the sign of β), and deformed triaxial for $\beta \neq 0, \gamma \neq 0$. Neglecting the higher-order terms and setting also $D = 0$, the analysis of Eq. (1) leads to a universal phase diagram [4,5] in the plane of two dimensionless parameters, $\tau = AC/B^2$ and ω/ω_c (with $\omega_c = 9B^{3/2}/16CR^{1/2}$).

For IBM, however, the above-outlined procedure meets a problem. In the standard geometric analysis [8–10], the quadrupole shape variables are introduced into the IBM as coefficients $a^{(2)}$ in the coherent (condensate) states,

$$|N, a_\mu\rangle = \frac{1}{\sqrt{\mathcal{N}}}\left(s^\dagger + \sum_{\mu=-2}^{+2} a_\mu d_\mu^\dagger\right)^N |0\rangle, \tag{2}$$

where N is the total number of s and d bosons and $\mathcal{N} = N!(1 + \sum_m |a_\mu|^2)^N$. The condensate states are used as trial wave functions for the zero-temperature variational procedure, the optimal ground-state configuration being obtained by minimization of the energy functional $\langle N, a_\mu|H|N, a_\mu\rangle$ (where H is the IBM Hamiltonian) with respect to a_μ. These variables form a second-rank tensor and the energy functional can be written as a function of only the principal-frame parameters β and γ, analogous to those introduced in the geometric model. However, for condensate states (2) with the coefficients satisfying the condition $a_{+\mu} = (-)^\mu a^*_{-\mu}$, resulting from Hermiticity of the Cartesian matrix a_{ij} associated with $a^{(2)}$, the average projection of the IBM angular momentum [7] $L = \sqrt{10}[d^\dagger \times \tilde{d}]^{(1)}$ to any direction is identically zero. Thus, the zero-temperature free energy $F = \langle H - \vec{\omega}\cdot\vec{L}\rangle_{T=0} = \langle N, a_\mu|H - \vec{\omega}\cdot\vec{L}|N, a_\mu\rangle$ remains unaffected by the cranking term.

The solution, as proposed by Schaaser and Brink [12,14], is to sacrifice the Hermiticity condition for the parameters $a^{(2)}$ and use more general condensate states that carry some angular momentum. The non-Hermitian matrix a_{ij} cannot be diagonalized so that $a_{\pm1}$ do not vanish in any frame. One can, nevertheless, deduce some natural constraints upon a_μ to be fulfilled in the intrinsic frame of the quadrupole deformation; these constraints follow from symmetry requirements under a certain discrete group of transformations [14]. As the direction of $\vec{\omega}$ can be chosen arbitrarily, we assume $\vec{\omega} = (\omega, 0, 0)$ for which the constraints read as $a_\mu = a_\mu^* = a_{-\mu}$. We therefore extend the standard Hill-Wheeler parametrization in the following way [15]:

$$a_0 = \beta\cos\gamma, \qquad \sqrt{2}\,a_{\pm1} = \beta\sin\gamma\sin\delta, \qquad \sqrt{2}\,a_{\pm2} = \beta\sin\gamma\cos\delta, \tag{3}$$

$\beta \in (-\infty, +\infty)$, $\gamma \in [0, \pi]$, and $\delta \in [-\pi/2, +\pi/2]$.

The geometric content of condensate states (2) with the substitution (3), $|N, a_\mu\rangle \equiv |N, \beta\gamma\delta\rangle$, can be read out from average components $\langle N, \beta\gamma\delta|Q_\mu^{(2)}|N, \beta\gamma\delta\rangle \equiv \langle Q_\mu^{(2)}\rangle$ of a physical quadrupole operator, usually chosen as $Q(\chi) = d^\dagger s + s^\dagger\tilde{d} + \chi[d^\dagger \times \tilde{d}]^{(2)}$. This gives $\langle Q_{\pm1}^{(2)}\rangle = 0$ and the nonzero components parametrized as $\langle Q_0^{(2)}\rangle/N = \tilde{\beta}\cos\tilde{\gamma}$ and $\sqrt{2}\langle Q_{\pm2}^{(2)}\rangle/N = \tilde{\beta}\sin\tilde{\gamma}$. The new parameters $\tilde{\beta}$ and $\tilde{\gamma}$, respectively, adopt the role of the old β and γ from Eq. (1) and can be interpreted in the same way as standard deformation parameters. Concrete expressions for the $Q(\chi)$ matrix elements can be found in Ref. [15]. The angular-momentum content of the $|N, \beta\gamma\delta\rangle$ states is given by the average projection of L onto the axis of rotation:

$$\langle L_x\rangle = \frac{2N\beta^2}{1+\beta^2}\underbrace{(\sqrt{3}\cos\gamma + \sin\gamma\cos\delta)\sin\gamma\sin\delta}_{l(\gamma,\delta)}. \tag{4}$$

We adopt the Hamiltonian from Refs. [6,15,16],

$$H(N, \eta, \chi) = \eta n_d - \frac{1-\eta}{N} Q(\chi)\cdot Q(\chi), \tag{5}$$

with $\eta \in [0, 1]$ and $\chi \in [-\sqrt{7}/2, +\sqrt{7}/2]$, where $n_d = d^\dagger\cdot\tilde{d}$ is the d-boson number operator. The shape-phase analysis of the Hamiltonian (5) [6,16,17] leads to the energy functional of the form (1), with $\omega = 0$ and T replaced by the control parameters η and χ, where the higher-order terms result just from an expansion of $(1 + \beta^2)^{-2}$ (see Refs. [6,16]). The phase diagram covers the extended Casten triangle [17] with vertices corresponding to the dynamical symmetries U(5), at $\eta = 1$, SU(3), at $(\eta, \chi) = (0, -\sqrt{7}/2)$, and $\overline{\mathrm{SU}(3)}$, at $(0, +\sqrt{7}/2)$. The O(6) symmetry is located at $\eta = \chi = 0$, which is the left end of the first-order phase-transitional line, $\chi = 0$, between prolate ($\chi < 0$) and oblate ($\chi > 0$) shapes. The first-order phase transition between deformed ($\eta < \eta_c$) and spherical ($\eta > \eta_c$) shapes takes place at $\eta = \eta_c = (4 + 2\chi^2/7)/(5 + 2\chi^2/7) + \mathcal{O}(N^{-1})$. At the intersection of both these curves, $(\eta, \chi) = (4/5, 0)$, the phase transitions become of the second order [6].

For $\omega \neq 0$, the zero-temperature free energy per boson in the classical limit, $F = \lim_{N\to\infty}\langle H(N, \eta, \chi) - \omega L_x\rangle/N$, can be evaluated [15] and yields

$$F(\eta, \chi, \omega; \beta, \gamma, \delta) = \frac{A(\eta, \omega; \gamma, \delta)\beta^2 + B(\eta, \chi; \gamma, \delta)\beta^3 + C(\eta, \chi, \omega; \gamma, \delta)\beta^4}{(1+\beta^2)^2}, \tag{6}$$

with

$$A = \eta - 4(1-\eta)(1 - \sin^2\gamma\sin^2\delta) - 2\omega l(\gamma, \delta), \tag{7}$$

$$B = 2\sqrt{2/7}\chi(1-\eta)[2\cos3\gamma + (7\cos\gamma - \sqrt{3}\sin\gamma\cos\delta)\sin^2\gamma\sin^2\delta], \tag{8}$$

$$C = \eta - \tfrac{2}{7}\chi^2(1-\eta)[1 - (\cos^2\gamma + 3\sin^2\gamma\cos^2\delta - \sqrt{3}\sin2\gamma\cos\delta)\sin^2\gamma\sin^2\delta] - 2\omega l(\gamma, \delta). \tag{9}$$

These expressions do not reproduce the form in Eq. (1). In particular, the dependence on ω is not quadratic, but linear. Indeed, with a non-Hermitian tensor $a^{(2)}$, there obviously *is* a possibility to build a rotational invariant of the first order in ω; it reads as $\varepsilon_{ijk}a_{jk}\omega_i$ (where ε_{ijk} is the Levi-Civita tensor). This leads to substantial modifications of the phase diagram following from Eqs. (6)–(9) in comparison with the analysis of Refs. [3–5].

With the form (6), the usual condition $A > 0$ for a minimum at $\beta = 0$ (the spherical phase) must be rewritten as $\min_{(\gamma,\delta)} A > 0$. This yields $\eta \geq 4/5$ and $\omega \leq \omega_c(\eta)$, where $\omega_c(\eta)$ is a critical frequency for a given η (one can equivalently get η_c as a function of ω). The spherical-deformed phase diagram is depicted in Fig. 1. When crossing the transitional point $\eta_c(\omega)$ at $\omega > 0$, the deformed phase immediately occurs in the triaxial form. This conforms with the observation [15] that within the IBM the triaxiality sets on with any small cranking frequency, which is valid also for the previous analyses [4,5]. (Note that with increasing ω the shape passes intermediate axisymmetric configurations as $\tilde{\gamma}$ crosses multiples of $\pi/3$.) However, a substantial difference compared to older results concerns cranking in the spherical case. Whereas the phase diagram in Refs. [4,5] implied that a spherical shape became deformed (oblate axisymmetric or triaxial) under anyhow small rotation, the present analysis shows that a certain critical rotation is needed to trigger the crossover to the deformed phase.

Note that the critical-frequency curve in Fig. 1 indicates where the local $\beta = 0$ minimum appears (disappears) and not where it becomes the global minimum. The transition between spherical and deformed phases being of the first order for $B \neq 0$ (i.e., for $\chi \neq 0$), the free energy develops a double-minimum form in the vicinity of the transitional point (a region of phase coexistence). At $\chi = 0$, however, B vanishes and the minimum at $\beta = 0$ coalesces with the one at $\beta \neq 0$ on the phase separatrix. Thus, the transition becomes of the second order at $\chi = 0$ and does not have coexisting phases. By analyzing the dependence in Eq. (6) in a close vicinity of the second-order phase separatrix, one derives the critical exponent describing the behavior of the order parameter

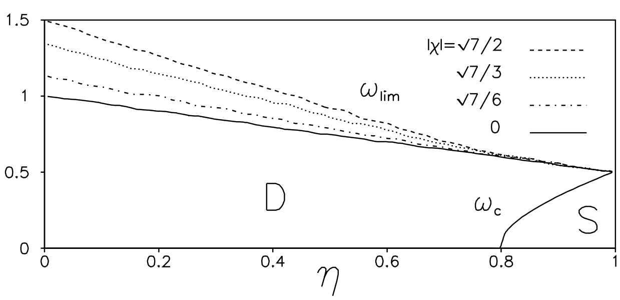

FIG. 1. The phase diagram of cranked IBM for the Hamiltonian (5). Spherical (S) and deformed (D) phases are separated by the critical frequency curve $\omega_c(\eta)$. The limiting frequency $\omega_{\rm lim}(\eta, \chi)$ is shown for given values of χ.

β immediately after the transition to the deformed phase. It is 1/2, the same as in the static case [6], in agreement with the general prediction of the Landau theory.

In fact, the $\chi = 0$ plane for $\omega > \omega_c(\eta)$ marks another shape-phase transition. Since the sign inversion of χ in Eq. (6) is compensated by the sign inversion of β, the global minimum of F jumps from β_0 to $-\beta_0$ when crossing the $\chi = 0$ plane. This change induces the sign inversion of the average quadrupole-tensor components and thus also $\tilde{\beta}_0 \rightarrow -\tilde{\beta}_0$. So the $\chi = 0$ first-order phase transition can be interpreted as a transition between prolatelike and oblatelike shapes, although the axial symmetry (and thus also the $\tilde{\gamma}$ softness at the transitional point) is the case only at $\omega = 0$.

It should be stressed that explicit analyses of the order of shape-phase transitions in the IBM without cranking were presented already in Refs. [9,10], and more recently also in Ref. [18]. The latter paper exploits the formalism of the catastrophe theory to determine all phase-separating curves in the two-dimensional essential parameter space of the static IBM, and for the first time contains an explicit discussion of the prolate-oblate phase transition. The present Letter directly extends these results to the cranking framework. As emphasized in Ref. [6], the isolated point of second-order phase transition between spherical and deformed prolate or oblate shapes in the static IBM is located at the intersection of three lines of first-order phase transitions. Indeed, in case of no additional symmetry constraints eliminating the B term in Eqs. (1) or (6), the Landau theory [1,2] shows that only such (and higher) intersections can host continuous (second-order) phase transitions between $\beta = 0$ and $\beta \neq 0$ phases. In the present case with cranking, the phase "diagram" has three dimensions and the second-order phase transitions are situated along a *curve* in intersection of three *surfaces* of first-order transitions.

While A and B in Eq. (6) are crucial for which phase the equilibrium solution belongs to, the value of C decides whether the solution exists or not. Because of the denominator in Eq. (6), C does not have to be positive as in Eq. (1)—in the present case the free energy does not diverge and C is its asymptotic value for $\beta \rightarrow \infty$. However, if C is, for some γ and δ, less than the minimal value F_0 of free energy obtained by minimization within finite β, the system cannot equilibrate but develops an infinite deformation. If stable solutions exist for some fixed values of η and χ, at low cranking frequencies, an increase of ω can lead to the situation when eventually $\min_{(\gamma,\delta)} C < F_0$. This is illustrated in Fig. 2.

It is not difficult to see that this scenario is connected with the finiteness of angular momentum in the IBM. Indeed, the maximum angular momentum with N bosons of the s and d type is $2N$, which is also the upper bound, $\beta \rightarrow \infty$, of $\langle L_x \rangle$. It can be shown that γ and δ that minimize C (for any η and χ) coincide with the values that maximize $l(\gamma, \delta)$ in Eq. (4) (yielding $l = 1$). In

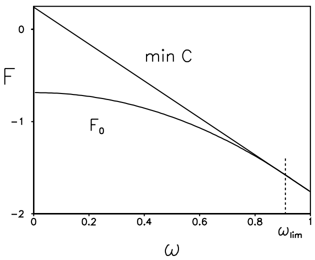

FIG. 2. Crossing of the $F_0 = \min_{(\beta,\gamma,\delta)} F$ curve (lower) with $\min_{(\gamma,\delta)} C$, see Eq. (6), determines the limiting frequency $\omega_{\lim}$. Calculation for $\eta = 1/2$ and $\chi = \pm\sqrt{7}/2$.

particular, we have $\min_{(\gamma,\delta)} C = \eta - 2\chi^2(1-\eta)/7 - 2\omega$. Thus, when the min$C$ line in Fig. 2 crosses the F_0 curve, the angular momentum reaches its maximal value. Since a further increase of ω would be unphysical, the divergence of solutions above the limiting frequency $\omega_{\lim}(\eta, \chi)$ is not surprising. The η dependence of the limiting frequency for some values of χ is shown in Fig. 1. We see that at $\eta = 1$ the critical and limiting frequencies coincide at the value $\omega_c = \omega_{\lim} = 1/2$. As a consequence, the U(5) limit does not exhibit a transition to the deformed phase for any physical ω. Viewed from the $\chi = 0$ plane, the point $(\eta, \omega) = (1, 1/2)$ is a critical point [1] terminating the curve of second-order phase transitions.

It is interesting to realize that the phase structure of the cranked IBM qualitatively resembles the situation known from physics of superconductivity. In this analogy, the superconducting and normal states are associated with the $\beta = 0$ and $\beta \neq 0$ phases, respectively. Indeed, under an increasing external magnetic field (equivalent to ω), the superconducting system exhibits a phase transition to the normal state, characterized by the onset of nonzero internal field B [19] (the transition is discontinuous or continuous for type-I or type-II superconductors, respectively). The phase diagram is similar as in Fig. 1 with η replaced by $(1 - T)$ (where T is absolute temperature). Similar behavior is predicted also for the pairing transition in the hot rotating nuclei [20] (although with both first- and second-order transitions on the same phase separatrix). The link between superconducting and normal phases, on one side, and the spherical and deformed phases, on the other, is suggested also by the geometric analysis of the fermion dynamical-symmetry model [21,22]. There, also, the corresponding phase transitions can be of the first or second order, in connection with the γ-rigid or γ-soft forms of the equilibrium solution. The said model was originally considered as the fermionic framework for the IBM and it seems, therefore, that the present cranking results probe the nature of the IBM phases.

There is no doubt that the standard shape-phase analysis including cranking, as presented in Refs. [4,5], is applicable in realistic situations. However, the similarity of the results outlined in this Letter to those from other fields suggests that the present alternative method of analysis is not just a mathematical curiosity. The key point leading to the deviation from the standard method is the fact that with s and d bosons one has only limited possibilities of building the angular-momentum operator and condensate states. It would be interesting to learn whether the use of more sophisticated interacting boson or boson-fermion models in the cranking framework would lead back to the results from Ref. [4,5].

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