Topological Order in the Insulating Josephson Junction Array

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We propose a Josephson junction array which can be tuned into an unconventional insulating state by varying external magnetic field. This insulating state retains a gap to half-vortices; as a consequence, such an array with nontrivial global geometry exhibits a ground state degeneracy. This degeneracy is protected from the effects of external noise. We compute the gaps, separating higher energy states from the degenerate ground state, and we discuss experiments probing the unusual properties of this insulator.

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A device implementing an ideal quantum computer would be a very interesting object from a physics point of view: it is a system with exponentially many degenerate states ($\mathcal{N}\sim 2^K$, $K \sim 10^4$) and extremely low decoherence rate [1]. The latter implies that the coupling of these states to the noise field produced by environment should be very small; more precisely, all states are coupled to the external noise in exactly the same way: $\langle n|\hat{O}|m\rangle = O_0\delta_{mn} + o(e^{-L})$, where $\hat{O}$ is an operator of the physical noise and L is a parameter such as the system size that can be made as large as desired. This condition is naturally satisfied [2] if degenerate states are distinguished only by a nonlocal topological order parameter [3,4]. In previous work [5] we proposed a superconducting Josephson junction array that, in addition to superconducting long range order, acquires a topological order parameter for the arrays with nontrivial geometry. The goal of this Letter is to identify the Josephson junction array with the ground state characterized *only* by the topological order parameter and no other long range order of any other type (in local variables) and which has a gap to all other (nontopological) excitations. Because such an array has neither superconducting phase stiffness nor gapless excitations, it cannot carry an electric current; thus, in the following we call it topological insulator. While arrays with similar global properties were proposed and studied in a recent work [6], the arrays discussed here have a number of important practical advantages which makes them more feasible to build and measure in the laboratory.

In physical terms, the topological superconductor appearing in the array [5] is a superfluid of $4e$ composite objects. The topological degeneracy of the ground state arises because $2e$ excitations have a gap. Indeed, in such a system with the geometry of an annulus, one extra Cooper pair injected at the inner boundary can never escape it; on the other hand, it is clear that two states differing by the parity of the number of Cooper pairs at the boundary are practically indistinguishable by a local measurement.

Generally, increasing the charging energy in a Josephson junction array makes it an insulator. This transition is due to an increase of phase fluctuations in the original array and the resulting appearance of free vortices that form a superfluid of their own. The new situation arises in a topological superconductor because it allows half-vortices. Two scenarios are now possible. The "conventional" scenario would involve condensation of half-vortices since they are conjugate to $4e$ charges. In this case, we get an insulator with elementary excitations carrying charge $4e$. An alternative is condensation of full vortices (pairs of half-vortices) with a finite gap to half-vortices. In this case, the elementary excitations are charge $2e$ objects. Similar fractionalization was discussed in the context of high-T_c superconductors in [7,8] and in the context of spin or quantum dimer systems in [9–13]. Such an insulator acquires interesting topological properties on a lattice with holes because each hole leads to a new binary degree of freedom which describes the presence or absence of a half-vortex. The energies of these states are equal up to corrections which vanish exponentially with the size of the holes. These states cannot be distinguished by local measurements and have all the properties expected for a topological insulator. They can be measured, however, if the system is adiabatically brought into the superconductive state by changing some controlling parameter.

In this Letter, we propose a modification of the array [5] that provides such a controlling parameter and, at the same time, allows us to solve the model and compute the properties of the topological insulator. The main physical idea is to construct a system where the kinetic energy gain of the half-vortex $t_{\rm hv}$ is *parametrically* smaller than the bandwidth of the full vortex $t_{\rm fv}$, while their respective potential energies $W_{\rm hv}$ and $W_{\rm fv}$ are of the same order and satisfy $t_{\rm hv} \ll W_{\rm hv} \sim W_{\rm fv} \ll t_{\rm fv}$. Under these conditions, Bose condensation of full vortices is energetically favorable, whereas half-vortices stay uncondensed and have a finite excitation gap. This is precisely the "fractionalized insulator" described above.

The array is shown in Fig. 1; it contains rhombi with junctions characterized by Josephson and charging energies $E_J > E_C$ and weak junctions with $\epsilon_J \ll \epsilon_C \ll E_C$. Each rhombus encloses half of a flux quantum leading to an exact degeneracy between the two states of opposite chirality of the circulating current [5,14]. This degeneracy is a consequence of the symmetry operation which combines the reflection about the long diagonal of the rhombus and a gauge transformation needed to compensate the change of the flux $\Phi_0/2 \rightarrow -\Phi_0/2$. This gauge transformation changes the phase difference along the diagonal by π. This Z_2 symmetry implies the conservation of the parity of the number of pairs at each site of the hexagonal lattice and is the origin of the Cooper pair binding. We assume that each elementary hexagon contains exactly k weak junctions. The important parameter is the number of weak junctions that one needs to cross in the elementary loop. Qualitatively, a value $k > 1$ ensures that it costs a little to put a vortex in any hexagon.

Practically, it is difficult to vary the ratio of the capacitance to the Josephson energy, so weaker Josephson contact usually implies larger Coulomb energy. This can be avoided if weak contact is made from a Josephson junction loop frustrated by magnetic field. The charging energy of this system is half the charging energy of the individual junction, while the effective Josephson junction strength is $\epsilon_J = 2\pi(\delta\Phi/\Phi_0)E_0$, where E_0 is the Josephson energy of each contact and $\delta\Phi = \Phi - (\Phi_0/2)$ is the difference of the flux from half-flux quanta.

Assume that ϵ_J sets the lowest energy scale in this problem (the exact condition will be discussed below). The state of the array is controlled by discrete variables $u_{ab} = 0, 1$, which describe the chiral state of each rhombus, and by continuous phases ϕ_{ab} that specify the state of each weak link (here and below a, b denote the sites of hexagonal lattice). If Josephson coupling $\epsilon_J \equiv 0$, different islands are completely decoupled and potential energy does not depend on discrete variables u_{ab}. For small ϵ_J, we can evaluate its effect in the perturbation theory:

$$V(u) = -W \cos\left(\pi \sum_{hex} u_{ab}\right), \qquad W = \frac{k^k}{k!}\left(\frac{\epsilon_J}{8\epsilon_C}\right)^{k-1}\epsilon_J. \tag{1}$$

This potential energy lowers the energy of classical configurations of u_{ab} that satisfy the constraint $\sum_{hex} u \equiv 0 \bmod 2$ but does not prohibit the ones with $\sum_{hex} u \equiv 1 \bmod 2$.

Consider now the dynamics of discrete variables. Generally, two types of tunneling processes are possible. In the first type, the phase changes by π across each of the three rhombi that have a common site. This is the same process that gives the leading contribution to the dynamics of the superconducting array [5]; its amplitude is given

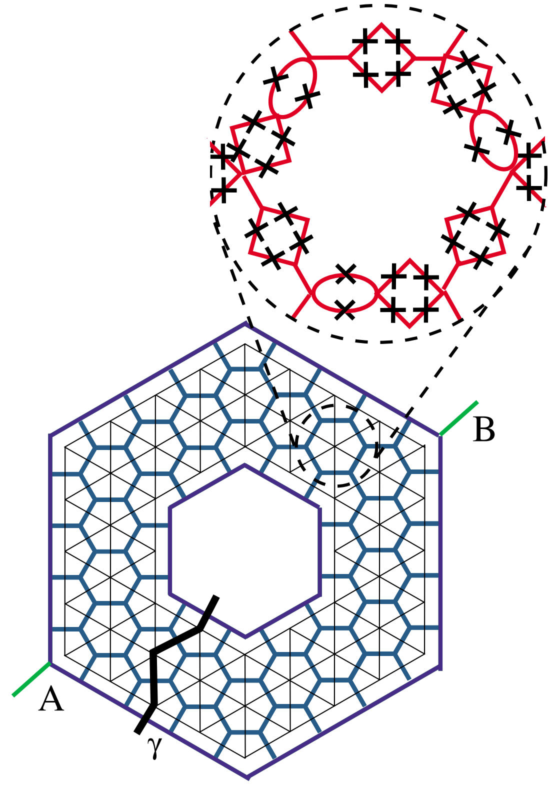

FIG. 1 (color online). Schematics of the array. The main figure: global structure of the array. Discrete variables controlling the low energy properties are defined on the links of the hexagonal lattice. Generally, the lattice might have K big holes; here we show a $K = 1$ example. Zoom in: Each inner bond of the lattice contains a rhombus made of four Josephson junctions; some bonds also contain an effective weaker link made of two Josephson junctions so that each hexagon of the lattice contains $k = 3$ such links. The flux through each rhombus is half-flux quanta $\Phi_0/2$; the flux through a loop constituting a weak link is close to half-flux quanta $\Phi = \Phi_0/2 + \delta\Phi$. The boundary of the lattice contains rhombi and weak links so that each boundary plaquette has the same number k of weak links as the bulk hexagon.

(in the quasiclassical approximation) by

$$r \approx E_J^{3/4} E_C^{1/4} \exp(-3S_0), \qquad S_0 = 1.61\sqrt{E_J/E_C}. \tag{2}$$

In the second type of process, the phase changes across one rhombus and across one weak junction. Because the potential energy of the weak junction is assumed to be very small, the main effect of the weak junction is to change the kinetic energy. The total kinetic energy for this process is the sum of the terms due to the phase across the rhombi and across the weak link. Assuming that these phase variations are equal and opposite in sign, the former is about $E_C^{-1}\dot{\phi}^2$ while the latter is $\epsilon_C^{-1}\dot{\phi}^2$, so the effective charging energy of this process is $\tilde{E}_C = (E_C^{-1} + \epsilon_C^{-1})^{-1}$. For $\epsilon_C \ll E_C$ this charging energy is small and such a process is suppressed. Thus, in these conditions the dominating process is the simultaneous flip of three rhombi as in the superconducting case. In the following, we restrict ourselves to this case. Further, we assume that $r \gg W$ so that in the leading order one can neglect the potential energy compared to the kinetic energy corresponding to the flip of three rhombi. As W is increased by turning on ϵ_J, the continuous phase ϕ_{ab} orders and the transition into the superconducting state happens at $\epsilon_J \sim \epsilon_C$. At larger ϵ_J, W becomes ϵ_J and with a further increase of ϵ_J, for $\epsilon_J \gg r$ vortices completely disappear from the low energy spectrum and the array becomes equivalent to the one studied in [5].

The low energy states are the ones that minimize the kinetic energy corresponding to simultaneous flip processes:

$$H_T = -r \sum_a \prod_{b(a)} \tau^x_{ab}. \tag{3}$$

Here $b(a)$ denote the nearest neighbors of site a, τ^x_{ab} is the operator that flips discrete variables u_{ab}, and r is given by (2). The states minimizing this energy satisfy the gauge invariance condition

$$\prod_{b(a)} \tau^x_{ab}|\Psi\rangle = |\Psi\rangle. \tag{4}$$

The Hilbert space of states that satisfy the condition (4) is still huge. If all weaker terms in the Hamiltonian are neglected, all states that satisfy (4) are degenerate. These states can be visualized in terms of half-vortices positioned on the sites of the dual lattice, i, j. Indeed, a convenient way to describe different states that satisfy (4) is to note that operator $\prod_{b(a)} \tau^x_{ab}$ does not change the value of $\sum_{j(i)} u \bmod 2$. Thus, one can fix the values of $\sum_{j(i)} u_{ij} = v_i$ on all plaquettes i and impose the constraint (4). In physical terms the binary values $v_i = 0, 1$ describe the positions of half-vortices on dual lattice. This degeneracy between different states is lifted when the subdominant terms are taken into account. The main contribution to the potential energy of these half-vortices comes from (1); it is simply proportional to their number. The dynamics of these vortices is due to the processes in which only one rhombus changes its state and the corresponding flip of the phase across the weak junction. The amplitude of this process is

$$\tilde{r} \approx E_J^{3/4} \tilde{E}_C^{1/4} \exp(-\tilde{S}_0), \qquad \tilde{S}_0 = 1.61\sqrt{\frac{E_J}{\tilde{E}_C}}.$$

The effective Hamiltonian of half-vortices is

$$H_v = -\tilde{r} \sum_{(ij)} \sigma^x_j \sigma^x_i - W \sum_i \sigma^z_i, \tag{5}$$

where operators $\boldsymbol{\sigma}_i$ act in the usual way on the states with/without half-vortices at plaquette i and the first sum runs over adjacent plaquettes (ij). This Hamiltonian describes an Ising model in a transverse field. For small $W/\tilde{r} < \lambda_c \sim 1$ its ground state is "disordered": $\langle \sigma^z \rangle = 0$ but $\langle \sigma^x \rangle \neq 0$, while for $W/\tilde{r} > \lambda_c$ it is "ordered": $\langle \sigma^z \rangle \neq 0$, $\langle \sigma^x \rangle = 0$. The critical value of transverse field is known from extensive numerical simulations [15]: $\lambda_c \approx 4.6 \pm 0.3$ for triangular lattice. The disordered state corresponds to the liquid of half-vortices, while in the ordered state the density of free half-vortices vanishes; i.e., the ground state contains an even number of half-vortices so the total vorticity of the system is zero. To prove this, we start from the state $|\uparrow\rangle$ which is the ground state at $\tilde{r}/W = 0$ and consider the effect of $\tilde{r}\sum_{(ij)} \sigma^x_j \sigma^x_i$ in perturbation theory. Higher energy states are separated from the ground state by the gap W so each order is finite. Further, in each order operator $\sigma^x_j \sigma^x_i$ create two more half-vortices proving that the total number of half-vortices remains even in each order. The states with an odd number of half-vortices have a gap $\Delta(\tilde{r}/W)$ which remains nonzero for $W/\tilde{r} > \lambda_c$.

In terms of the original discrete variable defined on the rhombi, the Hamiltonian (5) becomes

$$H_u = -\tilde{r} \sum_{(ij)} \tau^x_{ij} - W \sum_i \prod_{j(i)} \tau^z_{ij}, \tag{6}$$

where operators act on the state of each rhombus. This Hamiltonian commutes with the constraint (4); it is, in fact, the simplest Hamiltonian of the lattice Z_2 gauge theory. The disordered regime corresponds to a confined phase of this theory, leading to elementary $4e$ charge excitations and the ordered regime to the deconfined one.

Consider now the system with nontrivial topology, e.g., a hole. In this case, the set of variables v_i is not sufficient to determine uniquely the state of the system; one has to supplement it by the variable $v_0 = \sum_L u_{ij}$ where sum is taken over a closed contour L that goes around the hole. Physically, it describes the presence/absence of the half-vortex in the hole. The effective Hamiltonian of this additional variable has only a kinetic part because presence or absence of half-vortex in a hole which has l weak links in its perimeter gives potential energy $W_0 = c\epsilon_J(\epsilon_J/\epsilon_C)^l$ which is exponentially small for $l \gg 1$. The kinetic part is similar to other variables: $H_0 = -\tilde{r}\sum_{i \in I} \sigma^x_i \sigma^x_0$, it describes a process in which half-vortex jumps from the hole into the inner boundary I of the system. In the state with $\langle \sigma^z \rangle \neq 0$ this process increases the energy of the system by $\tilde{W}(W/\tilde{r})$ [$\tilde{W}(0) = W$ and $\tilde{W}(\lambda_c) = 0$]. In the state with $\langle \sigma^x \rangle \neq 0$ it costs nothing. Thus, the process in which a half-vortex jumps from the hole into the system and another half-vortex exits into the outside region appears in the second order of the perturbation theory; the amplitude of this process is $t_v = \tilde{r}^2 \sum_{i \in I, j \in O} g_{ij}$ where I and O denote sites of the inner and outer boundaries and $g_{ij} = \langle \sigma^x_i (1/H - E_0) \sigma^x_j \rangle$ is the half-vortex tunneling amplitude from inner to outer boundaries. At small $\tilde{r}/W$, we estimate g_{ij} using the perturbation expansion in $\tilde{r}/W$: the leading contribution appears in $|i-j|$th order of the perturbation theory, thus $g_{ij} \propto (\tilde{r}/W)^{|i-j|}$. Thus, for small $\tilde{r}/W$ the tunneling amplitude of the half-vortex is exponentially small in the distance L from the outer to the inner boundary; we expect that it remains exponentially small for all $\tilde{r}/W < \lambda_c$, while for $\tilde{r}/W > \lambda_c$ it is of the order of $\tilde{r}^2/W$ which is significant.

Mathematically, the ground state degeneracy is described by the topological invariant $\mathcal{P} = \prod_\gamma \tau^x_{ij}$ (contour γ is shown in Fig. 1) which can take values ± 1. The same arguments as used for the topological superconductor (TS) in [5] show that any dynamics consistent with constraint (4) preserves $\mathcal{P}$. Formally, the properties of the topological insulator (TI) are dual to that of the topological superconductor. This duality maps condensate of $4e$ charges in TS to the condensate of 2π phase vortices in TI. Fluxons that are gapped charge $2e$ excitations in TS become gapped π phase vortices in TI; pseudocharges that are half fluxes with energy $\epsilon \sim E_J \log(L)$ in TS become charge $2e$ with $\epsilon = 2r$ in TI [16]; ground state degeneracy is described by the charge on the inner boundary mod $4e$ in TS and by the number of π vortices inside the hole mod 2 in TI; ground state splitting is $[(\delta\Phi/\Phi_0)(E_J/r)]^L$ in TS and $(\tilde{r}/W)^L$ in TI.

Note that at small $\tilde{r}/W \to 0$ the ground state of the Hamiltonian (6) satisfies the condition (4) and minimizes the second term in (6), i.e., satisfies the condition $\prod_{j(i)} \tau^z_{ij}|\Psi\rangle = |\Psi\rangle$; it can be explicitly written as $|0\rangle = \prod_i \frac{1}{2}(1 + \prod_{j(i)} \tau^z_{ij}) \prod_{kl} |\rightarrow\rangle_{kl}$. This state is a linear superposition $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ of the degenerates states with $\mathcal{P} = 1$ and $\mathcal{P} = -1$; it coincides with the ground state of discrete variables in the superconducting array. The orthogonal superposition of $\mathcal{P} = \pm 1$ states, $\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$, corresponds to the half-vortex inside the hole and has a much larger energy in the superconductive array.

The degenerate ground states in the insulating array can be manipulated in the same way as in the superconductor. Since physically these states correspond to the absence or presence of the half-vortex inside the hole, the adiabatic change of local magnetic field that drags one half-vortex across the system flips the state of the system, providing us with the implementation of the operator τ^x acting on the state of the qubit. Analogously, motion of elementary charge $2e$ around the hole changes the relative phase of the states with and without a half-vortex, providing us with the operator τ^z.

The signature of the topological insulator is the persistence of the trapped half flux inside the central hole (see Fig. 1) which can be observed by cycling magnetic field so as to drive the system back and forth between insulating and superconducting states. This trapping is especially striking in the insulator. Experimentally, this can be revealed by slowly driving the array into a superconducting state and then measuring the phase difference between opposite points such as A and B in Fig. 1. In the state with a half-vortex the phase difference is $\pi/2 + \pi n$, while it is πn in the other state. The πn contribution is due to the usual vortices that get trapped in a big hole. This slow transformation can be achieved by changing the strength of weak links using the external magnetic field as a control parameter. The precise nature of the superconductive state is not essential because phase difference π between points A and B can be interpreted as due to a full vortex trapped in a hole in a conventional superconductor or due to a π periodicity in a topological one which makes no essential difference. These flux trapping experiments are similar to the ones proposed for high-T_c cuprates [8,17] with a number of important differences: the trapped flux is half of Φ_0, the cycling does not involve temperature (avoiding problems with excitations) and the final state can be either conventional or topological superconductor.

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