Inductance of rf-Wave-Heated Plasmas

E. Farshi and Y. Todo

Theory and Computer Simulation Center, National Institute for Fusion Science, 322-6 Oroshi-cho, Toki, Japan

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The inductance of rf-wave-heated plasmas is derived. This inductance represents the inductance of fast electrons located in a plateau during their acceleration due to electric field or deceleration due to collisions and electric field. This inductance has been calculated for small electric fields from the two-dimensional Fokker-Planck equation as the flux crossing the surface of critical energy $mv_{\mathrm{ph}}^2/2$ in the velocity space. The new expression may be important for radio-frequency current drive ramp-up, current drive efficiency, current profile control, and so on in tokamaks. This inductance may be incorporated into transport codes that study plasma heating by rf waves.

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Using radio-frequency waves for the generation and sustainment of plasma current in tokamaks has been of considerable interest, both in theory and in experiment. Radio-frequency waves have proved capable of driving the current necessary for the stability of tokamak plasmas, without any contribution from the dc electric fields [1]. The noninductive current drive effect is due to the asymmetric decrease of the electron collision rate caused by resonant absorption of rf power. The same decrease also implies an enhancement of the electrical conductivity [2] that is measured in some experiments [3,4]. This induced conductivity shows a nonlinear behavior in large electric fields and can represent the plasma as an active element [5]. On the other hand, the rate of change of the current drive can be significant, not only because of runaway acceleration by the electric field, but also because of deceleration by collisions. This causes an additional plasma inductance which can be important in studying plasma heating by radio-frequency waves. This is an induced inductance caused by radio-frequency heating of plasma and it is deeply different from a change of internal inductance due to a change of radial current profile in radio-frequency heating regimes that is observed in experiments [6].

Let us begin by explaining a physical picture. First assume a current drive regime that electrons have been moved due to radio frequency. Then consider that a parallel electric field is applied in the opposite direction of the electron movement. Therefore, the electrons can be accelerated by this field to a high enough energy that they overcome the dynamic frictional force of collisions. If the velocity domain V is large, then an electron accelerated from the bulk of the distribution to the boundary may be deemed a runaway, because frictional forces are extremely unlikely to return this electron to the velocity domain V. This will arise when the electric field is large enough. If the parallel electric field is applied in the same direction as the electrons, the electrons can be decelerated due not only to collisions but also the electric field to the thermal energy. However, the rate of change of the current drive can be significant, not only because of the electron acceleration by the electric field, but also because of the electron deceleration by the electric field and collisions, calculated from the Fokker-Planck equation as the flux crossing the surface of critical energy $mv_{\mathrm{ph}}^2/2$ in the velocity space, resulting in a large rate of change of high-energy current carries. In short, this effect introduces a plasma inductance which can significantly affect the current drive regimes in toroidal devices.

To estimate this inductance, a two-dimensional Fokker-Planck equation has been used. Presently, we will find an analytical solution using a perturbation technique. This solution will provide a useful approximation for small electric fields. These small electric fields may accelerate or decelerate the electrons according to their direction relative to electron direction in the current drive. Since the electric field is small in our model, the electrons finally will be decelerated to the thermal energy due to collisions. In the case when the phase velocity is larger than runaway velocity ($v_{\mathrm{ph}} > v_r = \sqrt{m\Gamma/qE}$, where $\Gamma \equiv nq^4 \ln\Lambda/4\pi\epsilon_0^2 m^2$), the resonance electrons located in the plateau of the distribution function will be accelerated as runaway. This will occur in large electric fields. The more exact numerical solution for these large electric fields by considering the runaway possibility will be numerically done in future work [7].

Consider a uniform electron-ion plasma, initially at equilibrium. Since in rf current drive regimes the distribution function of the bulk electrons remains Maxwellian, the problem can be linearized by substituting $f = f_m + f_1$ into the Boltzmann equation. This is because most plateau electrons collide with bulk electrons rather than with each other, since there are so many bulk electrons. The Fokker-Planck equation that will occupy our attention may be written as

$$\frac{\partial}{\partial t} f_1 + \frac{qE}{m}\frac{\partial}{\partial v} f_1 - C(f_1) = -\frac{\partial}{\partial v}\cdot S - \frac{qE}{m}\frac{\partial}{\partial v} f_m - \left[\frac{\dot{n}}{n} + \left(\frac{\epsilon}{T} - \frac{3}{2}\right)\frac{\dot{T}}{T}\right] f_m, \tag{1}$$

where we neglect spatial derivatives, $f_m = n(m/2\pi T)^{3/2}\exp(-\epsilon/T)$, and $S(v, t)$ is wave-induced flux. Here q, m, n, and T are the electron charge, mass, number density, and temperature, respectively, and ϵ is the energy of an electron. Here we have used the homogenous plasma approximation that is generally adequate for the lower hybrid current drives, because trapped-electron effects are small. The notational convenience $C(f_1) = C(f_1, f_m) + C(f_m, f_1) + C(f_1, f_i)$ is the linearized collision operator. Initially, at high speed, the current carried by electrons is substantial; when they slow down, they carry much smaller current and, because they are colliding frequently by then, even this small current persists for only a very short time. Therefore, it is a very good approximation to assume that the collisions always take place in the high-velocity limit, $v \gg v_T$, where we can simplify

$$C(f) \equiv \Gamma\left[\frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v_T^2}{v}\frac{\partial f}{\partial v} + f\right) + \frac{(1+Z) - v_T^2/2v^2}{2v^3}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu}f\right], \quad (2)$$

where $\mu = v_\parallel/v$, $\Gamma \equiv nq^4 \ln\Lambda/4\pi\epsilon_0^2 m^2$, $v_T^2 = T/m$, and ϵ_0 is the dielectric constant of free space; $\ln\Lambda$ is the Coulomb logarithm and Z is the effective ion charge state.

Since in most promising current-generation methods the distribution function of the bulk particles remains Maxwellian, the problem can be linearized by putting $f = f_m(1 + h)$. As already noted, most applied problems do not require knowledge of the distribution function $f(v, t)$, since their only requirement is the knowledge of several moments of $f(v, t)$. The longitudinal density of current is expressed by

$$J(t) = \int d^3v f_m(v, t)h(v, t)qv_\parallel. \quad (3)$$

According to Ref. [8], in the direction of applying the adjoint method we first define a commutative operation on the two functions $h(v, \acute{t})$ and $g(v, \acute{t})$:

$$[h, g]_t \equiv \int_V d^3v \int_0^t h(v, t-\tau)g(v, \tau)d\tau. \quad (4)$$

We also introduce the notation

$$j(v, t, \acute{t}) = \int d^3\acute{v}\, g(\acute{v}, v, t, \acute{t})qv_\parallel. \quad (5)$$

The function $j(v, t, \acute{t})$ is the influence function for the moment J. The corresponding influence function $j(v)$ is the solution of the following equation:

$$\frac{\partial j}{\partial t} + f_m\frac{eE}{m}\frac{\partial j}{\partial v_\parallel} + C^*(f_m j) = 0, \quad (6)$$

where C^* is the operator adjoint to C and may be written as $C^* = C(f_m j)/f_m$. The current density J will be

$$J = \int_0^t d\acute{t}\int_V d^3v\left(S + \frac{eE}{m}f_m\vec{e}_\parallel\right)\cdot\frac{\partial j}{\partial v}. \quad (7)$$

Equation (6) shows time dependence of current with electric field. Note the difference between Eqs. (1) and (6) to understand the meaning of Eq. (6). The Boltzmann equation (1) describes the evolution of a group of electrons released at $t = 0$ at velocity v, but $j(v, t)$ in Eq. (6) gives the mean current carried by those electrons at time t later. This current is carried by f_1. How Eq. (6) works is easily seen by taking $v \gg v_r$ so that the electrons experience only the electric field. In the Boltzmann equation (1) the electrons have slowed down to $v - (\Gamma/v_r^2)t$ at time t. Correspondingly, in the adjoint equation (6), the initial condition $j_0 = j(t = 0)$ is transported in the reversed direction so that $j(v, t) = j_0(v - (\Gamma/v_r^2)t)$.

In Ref. [2], Eq. (6) has been simulated as a familiar Ohm's law relation $J = \sigma E$ for small electric fields. On the other hand, in Ref. [5], Eq. (6) has been simulated as $J = \sigma(E)$ so that conductivity is a nonlinear function of E and it has not been limited to small electric fields. In both references, by choosing a special stationary state solution, the inductance effect has been deleted. However, according to Eq. (6) the rate of change of current drive can be significant and must be considered. Here we simulate Eq. (6) as a simple RL circuit equation $L(di/dt) + Ri = V$ (where R, L, i, and V are resistance, inductance, current, and voltage, respectively) instead of $J = \sigma E$. In another form, it will be

$$(a^2/2R_0)L\frac{\partial J}{\partial t} + \frac{J}{\sigma} = E, \quad (8)$$

where σ is the conductivity of rf-wave-heated plasma that has been obtained in Refs. [2,5]. R_0 and a are the major and minor radii of the torus, respectively.

The initial condition of Eq. (6) is $j = qv_\parallel$. Since the electric field is small, after a long enough time the electrons will be decelerated to the bulk ($j \rightarrow qv_{\rm th} \approx 0$) because of collisions. The time it takes the fast electrons to decelerate to the bulk depends on the collision operator and may be written as $v^3/\Gamma(5 + Z)\ln(v_\parallel/v_T)$. We integrate Eqs. (6) and (8) from $t = 0$ to $\tau = v^3/\Gamma(5 + Z)\ln(v_\parallel/v_T)$. Accordingly, define

$$\chi(v) = \int_0^\tau dt j(v, t), \quad (9)$$

where χ represents the resultant displacement of a charge having an initial velocity v (see Ref. [8]) that is the solution of the following equation [5,8]:

$$f_m\frac{eE}{m}\frac{\partial\chi}{\partial v_\parallel} + C^*(f_m\chi) = -f_m q\mu v. \quad (10)$$

Taking a perturbation expanding

$$\chi = \chi_0 + E\chi_1 + E^2\chi_2 + \ldots, \quad (11)$$

we get to the lowest order

$$\frac{\Gamma}{v^2}\left[\frac{v_T^2}{v}\frac{\partial^2}{\partial v^2}-\left(1+\frac{v_T^2}{v^2}\right)\frac{\partial}{\partial v}+\frac{(1+Z)-v_T^2/v^2}{2v}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu}\right]\chi_0=-q\mu v. \quad (12)$$

Then the first-order equation becomes

$$C(f_m\chi_1)=-f_m\frac{e}{m}\left(\mu\frac{\partial}{\partial v}+\frac{1-\mu^2}{v}\frac{\partial}{\partial \mu}\right)\chi_0 \quad (13)$$

and, for higher orders,

$$C(f_m\chi_n)=-f_m\frac{e}{m}\left(\mu\frac{\partial}{\partial v}+\frac{1-\mu^2}{v}\frac{\partial}{\partial \mu}\right)\chi_{n-1} \quad (14)$$

from which we may solve Eqs. (12)–(14) asymptotically in v_T/v:

$$\chi_0=\frac{e}{\Gamma}\frac{\mu v^4}{5+Z}\left[1+\frac{9}{(3+Z)}\left(\frac{v_T}{v}\right)^2+\dots\right]\approx\frac{e}{\Gamma}\frac{\mu}{5+Z}v^4, \quad (15)$$

$$\chi_1=\frac{e^2}{3m\Gamma^2}\frac{v^6}{5+Z}\left[1+\frac{3\mu^2-1}{3+Z}+O\left(\frac{v_T}{v}\right)^2\right]\approx\frac{e^2}{3m\Gamma^2}\frac{(2+Z+3\mu^2)}{(5+Z)(3+Z)}v^6, \quad (16)$$

$$\chi_2\approx\frac{2e^3}{m^2\Gamma^3}\frac{(24+19Z+3Z^2)\mu+(9+Z)\mu^3}{(3+Z)(5+Z)(7+3Z)(9+Z)}v^8, \quad (17)$$

where we have assumed $v_T \ll v_{\mathrm{ph}}$. The approximated orders of χ obtained by computer may be found in Ref. [9].

Integrate Eq. (8) from $t=0$ to $\tau=v^3/\Gamma(5+Z)\times\ln(v_\parallel/v_{\mathrm{th}})$ to obtain the inductance as

$$L_i=-\frac{\frac{P_d}{q\mu v}\chi\frac{S\cdot(\partial\chi/\partial v)}{S\cdot(\partial\epsilon/\partial v)}-\sigma_H\int_0^\tau dtE(t)}{(a^2/2R_0)\frac{P_d}{q\mu v}\frac{S\cdot(\partial\chi/\partial v)}{S\cdot(\partial\epsilon/\partial v)}\sigma_H q\mu v}. \quad (18)$$

This relation is obtained for a narrow spectrum of waves that we can write $J/P_d=[S\cdot(\partial\chi/\partial v)]/[S\cdot(\partial\epsilon/\partial v)]$, where ϵ and P_d are the energy of electron and the absorbed power per unit, respectively. Here σ_H is hot conductivity of fast electrons that may be given by [2,5]

$$\sigma_H=P_d\frac{S\cdot(\partial\chi_1/\partial v)}{S\cdot(\partial\epsilon/\partial v)}. \quad (19)$$

This relation for lower hybrid waves can be written as

$$\sigma_H\approx P_d\frac{2e^2}{m^2\Gamma^2}\frac{(2+Z+3\mu^2)}{(5+Z)(3+Z)}v_{\mathrm{ph}}^4. \quad (20)$$

Therefore, the inductance of lower hybrid wave heated plasma will be

$$L_i\simeq-\frac{1+\left(\frac{e(2+Z+3\mu^2)}{3\mu m\Gamma(3+Z)}v_{\mathrm{ph}}^2\right)\left[E-\frac{3(5+z)\Gamma}{v_{\mathrm{ph}}^3}\int_0^\tau dtE(t)\right]}{P_d\frac{a^2e^2}{R_0m^2\Gamma}\frac{(2+Z+3\mu^2)}{(3+Z)}v_{\mathrm{ph}}}. \quad (21)$$

This relation shows the average induced inductance of fast resonance electrons in the time interval $0<t<\tau$ during their deceleration. The fast electrons also experience the usual inductance of tokamak plasma due to the torus property of the tokamak, $L=\mu_0R_0[\ln(8R_0/a)-\frac{7}{4}]$. Therefore, the total inductance of these fast electrons will be the summation of induced and usual inductances, $L_f=L+L_i$. By considering the inductance of bulk electrons as L_b, the total inductance of a tokamak plasma in radio-frequency heating regimes will be as $L_{\mathrm{tot}}=L_f\parallel L_b$; thus, inductances of fast and bulk electrons have been considered as parallel. It is the same as the resistances of fast and bulk electrons in Refs. [2,5]. In these references, the resistances of fast (R_H) and bulk (R_{sp}) electrons were considered as parallel (i.e., $\sigma_{\mathrm{tot}}=\sigma_H+\sigma_{\mathrm{sp}}$) but eliminated the induced inductance of fast electrons. In more exact calculation, it is better to consider two RL circuits as series (i.e., $R_fL_f\parallel R_{\mathrm{sp}}L_b$).

Let us investigate a sign of induced inductance. The induced inductance is negative in the case of $E=0$. It is evident from relation (21) that when electric field E is applied in the same direction as the electrons (i.e., E and μ have the same sign) the electrons can be decelerated due not only to collisions but also to the applied electric field that leads to a decrease of the inductance in relation (21). On the other hand, if electric field E is applied in the opposite direction as the electron movement (i.e., E and μ do not have the same sign), the electrons can be accelerated opposite of deceleration by collisions that lead to an increase of the inductance in relation (21). In this case, when electric field E reaches values larger than the critical limit, $[\frac{e(2+Z+3\mu^2)}{2\mu m\Gamma(3+Z)}(\ln(\frac{v_{\mathrm{ph}}}{v_{\mathrm{th}}})-\frac{2}{3})v_{\mathrm{ph}}^2]E>1$, the acceleration effect due to the electric field will be dominant compared to the deceleration effect due to collisions. Therefore, the sign of the inductance in relation (21) will be changed. This will occur in large electric fields but our solution is limited to $E\to 0$ that shows that relation (21) is correct only for small electric fields.

In the case of $E=0$, we have

$$L_{i,(E=0)}=-\frac{R_0\Gamma m^2(3+Z)}{P_da^2e^2(2+Z+3\mu^2)v_{\mathrm{ph}}}. \quad (22)$$

Therefore, it is seen from relation (22) that an inductance property exists even for $E=0$ but it will be significant in the presence of electric field. For example, in a tokamak with major parameters of $R_0=0.8m$, $a=0.12m$, $n=3\times10^{19}m^{-3}$, $P_{\mathrm{rf}}=100$ kW, $\eta=0.3$, $v_{\mathrm{ph}}=10^8/4$, $Z_{\mathrm{eff}}=3$, the induced inductance in the absence of electric field will be $L_i=-0.7\ \mu$H, but in the presence of electric field it will be more significant. By considering the usual inductance of these fast electrons in this tokamak as $L=2.2\ \mu$H, the total inductance of fast electrons due to radio-frequency current drive will be $L_f=L+L_i=1.5\ \mu$H. The time constant of decay for such resonance electrons in $E = 0$ may be given as $L/R = v_{\mathrm{ph}}^3/\Gamma(5 + Z)$. This parameter does not depend on the rf power P_d in contrast to the inductance relations (21) and (22) and also the conductivity relations (19) and (20) that depend on the rf power. In the presence of electric field, the time constant of fast electron decay will be dependent on E.

Let us review briefly the physical picture. Electrons absorbing wave energy and momentum may slow down either by colliding with the background plasma or by decelerating under the effect of the dc electric field. As an electron decelerates, it transfers a part of its kinetic energy to the bulk electrons with which it collides and the remainder of its kinetic energy to the electromagnetic field that decelerates it. The latter energy contribution appears as magnetic energy storage (i.e., $LI^2/2$), while the former contribution appears merely as heat. For small P_d, the induced inductance effect causes a negative inductance for fast electrons, which means that power flows from the field energy into the kinetic energy of resonant electrons because rf power is insufficient (or misdirected). When the absorbed rf power P_d increases, the induced inductance (L_i) of fast electrons also increases (i.e., $|L_i|$ decreases) which causes the total inductance of fast electron (L_f) to increase. In the case of $L_i = -L$, the total inductance of fast electrons will be zero (i.e., $L_f = 0$), which causes the conversion of all of the kinetic energy of fast electrons into heat. For high P_d, the induced inductance of fast electrons tends to zero, so that the total inductance of fast electrons will tend to L. In this regime, all of the kinetic energy of fast electrons will be converted to magnetic field energy. Note the dependency of induced inductance on density (Γ). When density increases, the inductance of fast electrons decreases, so that contributions of fast electrons to heat increase. This is because the electron is sensitive to collisions in high densities; therefore, collisions tend to slow down the electron, and electron kinetic energy will be converted to heat.

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[7] See EPAPS Document No. E-PRLTAO-90-040310 for a numerical solution for large electric fields using a finite element code. A direct link to this document may be found in the online article's HTML reference section. The document may also be reached via the EPAPS homepage (http://www.aip.org/pubservs/epaps.html) or from ftp.aip.org in the directory /epaps/. See the EPAPS homepage for more information.
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