

## Intermittency and Non-Gaussian Fluctuations of the Global Energy Transfer in Fully Developed Turbulence

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We address the experimentally observed non-Gaussian fluctuations for the energy injected into a closed turbulent flow at fixed Reynolds number. We propose that the power fluctuations mirror the internal kinetic energy fluctuations. Using a stochastic cascade model, we construct the excess kinetic energy as the sum over the energy transfers at different levels of the cascade. We find an asymmetric distribution that strongly resembles the experimental data. The asymmetry is an explicit consequence of intermittency and the global measure is dominated by small scale events correlated over the entire system. Our calculation is consistent with the statistical analogy recently made between a confined turbulent flow and a critical system of finite size.

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Intermittency is a well established feature of turbulent flows. It has been observed in a large number of experiments and incorporated in many models [1,2]. In this Letter we attempt to link the statistics of the fluctuations of the global energy consumption  $P_I$  in a turbulent flow to the intermittency of the energy transfer. Our motivation stems from the empirical observation [3] that the probability density function (PDF) of power consumption  $P_I$  for a confined turbulent flow driven at constant Reynolds number  $Re$  has a non-Gaussian form, asymmetric with an exponential tail [4,5], extremely similar to that observed for the order parameter fluctuations in a finite size equilibrium system at criticality. Very similar distributions can be observed for global quantities in other experimental [6–9] and model correlated systems [10]. These observations have generated recent interest in diverse aspects of global fluctuations, for systems both in and out of equilibrium [11–16]. We have previously argued that the observation of similar fluctuations in the global quantity for these radically different correlated systems implies an analogy at the statistical level [3]. Specifically, we have shown [17], using the low temperature phase of the 2D-XY model, that a key ingredient for the occurrence of such non-Gaussian fluctuations is the action of modes with a diverging distribution of spin wave stiffness across spatial scales. In order to apply these findings to the statistics of power injection in a confined turbulent flow, we consider turbulence in the framework of a stochastic cascade across scales. The flow state, at a statistical level, is viewed as a gas of independent fluctuations at all scales  $\ell$  between the injection and dissipative lengths,  $\eta$  and  $L$ . The variance of the fluctuations of these modes of energy transfer diverge at small scale, which is the well accepted feature of turbulence, called intermittency [1]. The divergence across the scales is analogous to that observed in the model critical system. In this Letter we therefore show, using a phenomenological model, that the observed non-Gaussian fluctuations of

the global power injection are a natural consequence of intermittency.

Experimental data, reproduced from Ref. [5], are shown in Fig. 1. The flow is driven by two concentric, counterrotating disks, with cylindrical geometry, rotating at constant frequency  $\Omega$ . The probability distribution for the fluctuations in the power,  $P_I(t)$ , needed to drive the flow is characterized by a negative skewness, with an approximately exponential tail for fluctuations below the mean and a much more rapid falloff for fluctuations above the mean. The energy balance is given by [4]

$$\frac{dE(t)}{dt} = P_I(t) - P_D(t), \quad (1)$$

where  $E(t)$  and  $P_D(t)$  are, respectively, the instantaneous kinetic energy and dissipation by viscous forces [2,4] of the confined fluid. A time average of Eq. (1) yields the condition of stationarity:  $\overline{P_I} = \overline{P_D} = M\epsilon_0$ , where we

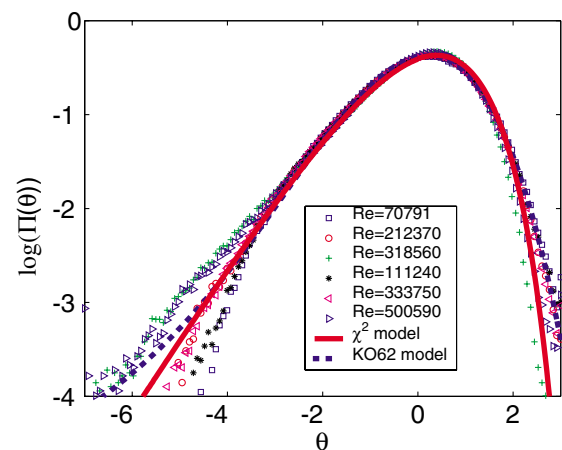


FIG. 1 (color online). PDF of global energy transfer for experimental data, taken from Pinton *et al.* [5] (symbols) and from cascade models of intermittency. The cascade models are the  $\chi^2$  model,  $Re = 10^5$  and  $\lambda = 2.0$  (solid line) and the log-normal model,  $Re = 10^5$  and  $\lambda = 1.08$  (dashed line).

define  $\epsilon_0$ , the power consumption of the flow per unit mass, and  $M$ , the total mass of the fluid.

In the absence of a microscopic description of the statistical properties of the flow, a phenomenological model is required to explain the observed fluctuations about this mean value: first, we propose that the fluctuations in  $P_I(t)$  are related to fluctuations in the internal kinetic energy of the flow

$$P_I(t) = M\epsilon_0 - \mathcal{E}(t). \quad (2)$$

Here,  $\mathcal{E}(t) = [E(t) - \bar{E}]/\tau$  is the difference between the instantaneous and mean kinetic energy, normalized by  $\tau$ , an integral time scale characteristic of the energy injection and of the response of the driving mechanism (inertia). That is, we propose that the torque required to shear a fluid at constant rate  $\Omega$  is a decreasing function of the internal kinetic energy of the fluid. We predict, therefore, that the constraint,  $\Omega = \text{const}$ , forces the fluctuations in  $P_I(t)$  to be a mirror reflection of the fluctuations of the internal energy of the flow. If constrained with constant torque one would consequently expect reversed power fluctuations with positive skewness, as also observed experimentally [18]. We do not suppose that mechanical structures drive the disks, simply that the engagement of the disks is reduced with increasing kinetic energy of the underlying flow. An excess of kinetic energy should be repartitioned over the entire flow and felt simultaneously at the two disks, which would then decrease their engagement in the fluid, independently of their rotation direction. This scenario provides a stabilizing loop through which the turbulent steady state is maintained, and it is consistent with experimental observation that the time variation of the power injected into the upper and lower disks displays positive correlations: power fluctuations of all amplitudes are felt quasimultaneously at both disks, despite their rotation in opposite directions [see Fig. 1(a) of Ref. [5]].

Second, we build  $\mathcal{E}(t)$  as the excess of transferred energy at time  $t$ , summed over contributions from all scales in the flow. For this we introduce  $\pi_\ell(\vec{r}, t)$ , the instantaneous real-space energy flux:

$$\pi_\ell(\vec{r}, t) = -\frac{1}{4}\vec{\nabla}_\ell \cdot [|\delta_\ell \vec{u}(\vec{r})|^2 \delta_\ell \vec{u}(\vec{r})], \quad (3)$$

which we average over the flow volume to obtain the energy flux at scale  $\ell$ ,

$$\pi_\ell(t) = \frac{1}{V} \int_V d^3r \pi_\ell(\vec{r}, t); \quad (4)$$

these variables could be computed in a direct numerical simulation. Our phenomenology concerns the fluctuations away from the steady state at all scales  $\ell$ :  $[\pi_\ell(t) - \epsilon_0]$ . We adopt a statistical point of view rather than a dynamical one, and we make the assumption that the  $(\pi_\ell - \epsilon_0)$  are statistically independent stochastic variables [19] (one should think of this as a nonlinear change of variables rather than a physical tree structure replacing the real flow). We then define the excess energy  $\mathcal{E}$  as the sum over

contributions from all scales:

$$\mathcal{E} = M \sum_\ell (\pi_\ell - \epsilon_0). \quad (5)$$

This is a strong approximation, whose justification rests with the results it produces, which are in striking agreement with experiment—see Fig. 1. Physically, our model implies that the energy injection at a given time takes into account the entire structure of the energy flow at that time. When  $\pi_\ell(t) > \epsilon_0$ , energy is built up at scale  $\ell$  and the overall energy excess  $\mathcal{E}(t)$  is the sum of such effects over the entire range of scales.

Equation (5), together with our assumptions, allows one to construct the PDF of  $\mathcal{E}$  as the convolution of probability densities for the elements  $(\pi_\ell - \epsilon_0)$ :

$$\Pi(\mathcal{E}) = \prod_{\ell}^{\infty} p_\ell(\pi_\ell - \epsilon_0), \quad (6)$$

where  $p_\ell(\pi_\ell)$  is the PDF for the fluctuations of  $\pi_\ell$ . Once the  $p_\ell(\pi_\ell)$  are chosen, the above expression allows for the numerical computation of the distribution for the global quantity. In fact, one can express the PDF for the normalized variable  $\phi = \frac{\mathcal{E} - \langle \mathcal{E} \rangle}{\sigma}$  as

$$\Pi(\phi) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik\phi} \Psi(k), \quad (7)$$

where the characteristic function  $\Psi(k)$  is the product of the characteristic functions of each of the microscopic distributions and the net variance  $\sigma^2$  is the sum of the microscopic variances. Then, using Eq. (2), the PDF for the fluctuations of the normalized power input ( $\theta = \frac{P_I - \bar{P}_I}{\sigma_{P_I}}$ ) is  $\Pi(\theta) = \Pi(-\phi)$ .

The phenomenology proposed here cannot be developed within the context of Kolmogorov's 1941 (K41) theory of turbulence [1], as it ignores fluctuations around mean values of energy injection and transfer. We therefore develop Eq. (5) in the context of Kolmogorov's refined similarity hypothesis [2,20]. It allows for anomalous scaling of the energy transfer at scale  $\ell$ :

$$\langle \pi_\ell^q \rangle \propto \epsilon_0^q (\ell/L)^{\tau(q)}. \quad (8)$$

The spectrum  $\tau(q)$  is related to the experimentally observed nonlinearity in the scaling exponents of the longitudinal velocity increments:  $\langle \delta_\ell u^q \rangle \propto \ell^{\zeta(q)} \propto \langle \pi_\ell^{q/3} \rangle \times (\ell/L)^{q/3}$  so that  $\zeta(q) = q/3 + \tau(q/3)$ .

In the following we use two different forms for the microscopic distributions, compatible with the experimentally determined exponents for the velocity structure functions  $\zeta(q)$ . We show that intermittency is necessary to obtain asymmetric non-Gaussian distributions with negative skewness. However, let us first return to some details of the model: the  $(L/\eta)^3$  correlated degrees of freedom implicated in the flow [2] are transformed into an ensemble of  $N$  scales. At each level the system is described with a resolution length scale  $\ell_n$  and the scales are separated by a ratio  $\lambda = (\ell_{n-1}/\ell_n)^3$  such that  $\lambda^N = (L/\eta)^3$ . The Reynolds number and the number of levels are related through the definition  $\text{Re} = (L/\eta)^{4/3} = \lambda^{4N/9}$ .

Note that the variable  $\lambda$  is a free parameter in the theory. In the original Kolmogorov-Obukhov theory (KO62) of intermittency [2], the energy cascade, being a multiplicative process with a large number of steps, is assumed to have log-normal statistics. In this case  $p_n(\log \pi_{\ell_n})$  is a Gaussian distribution whose variance can be that of KO62 or that proposed by Castaing *et al.* [21]. As one assumes scale invariance in KO62, the variance is  $\sigma_n^2 = \epsilon_0^2 [(\ell_n/L)^{\tau(2)} - 1]$ . This distribution leads to the quadratic spectrum  $\zeta(q) = q/3 + \mu q(3 - q)/18$ . The value of the intermittency parameter  $\mu \equiv -\tau(2)$  that fits best the experimental data is  $\mu = 0.21$  [22]. Another possibility, which gives a very good fit to the velocity intermittency exponents is a  $\chi^2$  distribution:

$$p_n(\pi_{\ell_n}) = N_{\ell_n} \pi_{\ell_n}^{\nu_{\ell_n}-1} \exp(-a_{\ell_n} \pi_{\ell_n}). \quad (9)$$

In this case, we require that the variance of  $p_n(\pi_{\ell_n})$  be that of KO62, in addition to the conditions of normalization and constant mean. The three conditions determine  $N_{\ell_n}$ ,  $a_{\ell_n}$ ,  $\nu_{\ell_n}$ . From the expression for the moments [ $\langle \pi_{\ell_n}^q \rangle = \epsilon_0^q \Gamma(q + \nu_{\ell_n}) / \nu_{\ell_n}^q \Gamma(\nu_{\ell_n})$ ] it is straightforward to show that the spectrum  $\tau(q)$  is that of KO62 in the limit  $\ell_n \rightarrow L$ . Even if  $\chi^2$  statistics does not strictly allow for scale invariance for  $\tau(q)$  and  $\zeta(q)$  in the inertial range, the corrections are very small. The main advantage of the  $\chi^2$  distribution is that it allows a straightforward calculation of the convolution product in Eq. (7):

$$\log \Psi(k) = ik\epsilon_0/\sigma - \sum_{n=1}^N \nu_{\ell_n} \log[1 + ik/(\sigma a_{\ell_n})]. \quad (10)$$

In Fig. 1 we show the experimental data published in Ref. [5] together with the PDF for  $\Pi(\theta)$ , calculated from Eq. (5), using both log-normal and  $\chi^2$  models. They both capture the essential features of the experimental data: (i) the fluctuations in  $\theta$  are strongly non-Gaussian, despite the high value of the Reynolds number, i.e., of the large number of contributions in the sum (5); (ii) the distribution is skewed towards negative values; (iii) the tails of the PDF towards negative values are almost exponential, although this seems better verified for the  $\chi^2$  microscopic distribution than for the log-normal distribution. This behavior is extremely robust against variations of the Reynolds number, at fixed  $\lambda$ . In this case, increasing the Reynolds number leads to an increase of the number of cascade steps  $N$  (as  $\log \text{Re}$ ) and, at the same time, to an increase of the asymmetry in the shape of the microscopic distribution at the smallest scales. In Fig. 2 we show the evolution of the distribution with  $\text{Re}$ , for  $\lambda = 2.0$ , for the  $\chi^2$  microscopic distributions. The curves vary only slowly with  $\text{Re}$ . The asymmetry becomes more pronounced as  $\text{Re}$  increases, as can be seen from the skewness,  $\gamma$ , which varies from  $-0.64$  to  $-0.97$  for  $\text{Re}$  varying from  $10^4$  to  $10^7$ . Hence, as in the experiment, there is no evidence that  $\Pi(\theta)$  will reduce to a Gaussian in the limit of infinite Reynolds numbers, and therefore

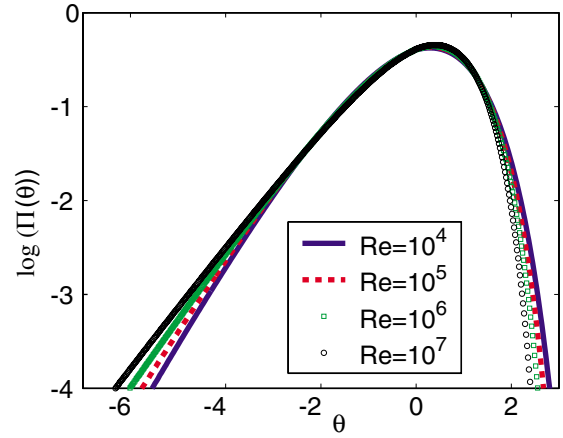


FIG. 2 (color online). PDF of global transferred energy for  $\text{Re}$  varying between  $10^4$  and  $10^7$  (legend). The microscopic distributions are  $\chi^2$  ones and  $\lambda = 2.0$ .

infinite  $N$ . Given the uncertainty in the experimental results and the crudeness of our model, this qualitative agreement seems extremely encouraging.

As  $\lambda$  is a free parameter in the problem, it has been chosen in Fig. 1 to give a good fit of the experimental curves for each class of microscopic distribution. For a fixed value of  $\text{Re}$ , varying  $\lambda$  does not change the shape of the global PDF. Using the  $\chi^2$  distribution, we observe a slight increase of the skewness, from  $-0.65$  to  $-0.83$  when  $\lambda$  is increased from 1.5 to 2 (for  $\text{Re} = 10^6$ ), a typical range of values used in cascade models [1].

However, there seems to be no *a priori* reason to fix  $\lambda$  and while we do not rule out a weak variation, we can fix a universal form for the PDF, as proposed in [3] by letting  $\lambda$  vary with  $\text{Re}$ . Support for this proposition comes from the evolution of the ratio  $\sigma/\overline{P}_I$  with  $\text{Re}$ . Experimentally, one observes a very slow decrease that may be fitted by a power law dependence with exponent  $\alpha \sim 0.3$  [5]. In our computations at fixed  $\lambda$ , we find the ratio to be slightly increasing, in disagreement with experiment. If instead of fixing  $\lambda$  we impose universality, we find a decreasing ratio, with a weak power law dependence with exponent  $\alpha \sim 0.1$ .

Here, the small scales play a major role due to the dispersion in the amplitudes and as in the finite size critical system [23], one might expect the global PDF to retain some characteristics of the microscopic measure, with the experimentally observed asymmetry reflecting the form of  $p_n(\pi_{\ell_n})$ . This is indeed what we observe and the symmetry of the global PDF is restored only if the individual elements are made to have equivalent weight. This is illustrated in Fig. 3, where we show a series of PDFs produced at fixed  $\text{Re}$  and  $\lambda$ , with  $\tau(2)$  varying between the experimental value, 0.21 and zero. As  $\tau(2)$  is reduced the PDF becomes more symmetric and, for  $\tau(2) = 0$  it is well represented by a Gaussian. This point is quite important: intermittency is absolutely necessary to obtain an asymmetric and non-Gaussian PDF for the

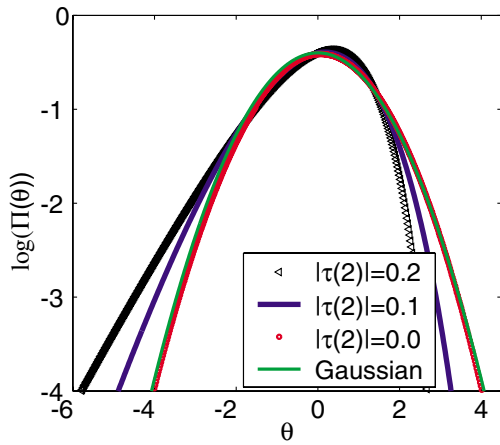


FIG. 3 (color online). Variation of the global PDF with the magnitude of the intermittency parameter  $|\tau(2)|$  for  $\text{Re} = 10^6$  and  $\lambda = 2.0$ . The microscopic distributions are  $\chi^2$  statistics.

global energy transfer. However this conclusion holds only for the PDF of energy transfer. In the case of pressure fluctuations Holzer *et al.* [24] have shown that intermittency is irrelevant since for the pressure one integrates  $\delta_l u^2$  which gives a dispersion of variance  $\langle \delta_l u^2 \rangle \sim (\epsilon_0 l)^{2/3}$  in the strict K41 description. We remark that, with the phenomenology proposed here, negative skewness is a direct consequence of kinetic energy fluctuations with positive skewness, as are observed in microscopic models with dissipation [12–14].

The importance of the small scale can be gauged by recalculating  $\mathcal{E}$ , excluding levels from either the top or the bottom of the cascade. We find that with removal of cascade levels near the dissipative scale the distribution rapidly approaches a Gaussian form, while removal of levels from the top of the cascade leaves the PDF essentially unchanged. The entire global measure is therefore seen to be influenced by a relatively small number of statistically independent cascade levels describing the system at small scales. If a universal PDF for the global measure is indeed an experimental reality, then the dispersion in amplitudes must diverge with the Reynolds number in such a way that the universal distribution is a “limit” function, valid even as  $\text{Re} \rightarrow \infty$ . These arguments lead us to a striking physical conclusion concerning the nature of a turbulent flow: given the huge number of small scale objects present in the flow [ $(L/\eta)^3 = \text{Re}^{9/4} \sim 10^9$  for  $\text{Re} = 10^4$ ]; if the PDF for the global measure is non-Gaussian, it can mean only that these objects are strongly correlated in space and time. This point seems consistent with recent measurements of long time temporal correlations in the dynamics of Lagrangian tracer particles in a fully turbulent flow [25].

We finally address the point that the PDF of the injected power has a Gaussian shape for open (nonconfined) flows [4]. The distribution of the global quantity thus depends on the overall size of the flow, although it is known that small scale intermittency characteristics do

not; for instance, the scaling exponents of the longitudinal velocity increments are identical in open or confined flows [22]. We propose that in nonconfined flows several uncorrelated cascades occur simultaneously. In this case, the probability for the overall energy transfer is given as the convolution of  $\mathcal{N}$  functional forms of the type  $\Pi(\theta)$  computed above. This tends rapidly to a Gaussian. Experimentally, this could be tested by changing continuously the ratio of the disk to cylinder radius in a “washing-machine” experimental setup.

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