Aging in Subdiffusion Generated by a Deterministic Dynamical System

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We investigate aging behavior in a simple dynamical system: a nonlinear map which generates subdiffusion deterministically. Asymptotic behaviors of the diffusion process are described using aging continuous time random walks. We show how these processes are described by an aging diffusion equation which is of fractional order. Our work demonstrates that aging behavior can be found in deterministic low dimensional dynamical systems.

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Aging behavior is found in complex dynamical systems like spin glasses, glasses, polymers [1], and in random walks in random environments [2]. These systems usually include many interacting subunits and are disordered, and their dynamics is driven by noise. In this Letter I will demonstrate that aging can be found also in low dimensional deterministic systems. Specifically, I will show that diffusion generated by one-dimensional maps exhibits aging and that statistical properties of the corresponding trajectories can be analyzed using aging continuous time random walks (ACTRW), introduced by Monthus and Bouchaud [3] as a simple phenomenological model of aging dynamics in glasses (see also [4] and references therein).

There exist several methods to investigate aging. One method is to start a dynamical process at time $t = -t_a$, then at time $t = 0$ add a small perturbation to the system. One eventually measures the response at some time $t > 0$. Alternatively, one can measure correlation functions between physical quantities at time $t = 0$ and time t, after aging the system in the interval $(-t_a, 0)$. I use the latter approach. A system exhibits aging if its dynamical properties depend on t and t_a even in the limit when both are long. Of course, many systems do not exhibit aging, namely, when $t > \tau$, where τ is a characteristic time scale of the problem, dynamical properties of the system are independent of the aging time t_a.

In many cases trajectories generated by deterministic systems, such as low dimensional Hamiltonians or maps, are highly irregular. For an observer, these trajectories seem to be generated by a stochastic rather than a deterministic mechanism. Hence analysis of chaotic trajectories generated deterministically is often based on random walk concepts [5–7]. It is well known [8] that both conservative and dissipative deterministic systems may generate normal ($\alpha = 1$) or anomalous ($\alpha \neq 1$) diffusion for a coordinate x,

$$\langle x^2 \rangle \sim t^{\alpha}, \tag{1}$$

where the average is over a set of initial conditions. The anomalous behavior is due to long trapping events in the vicinity of unstable fixed points [9,10] or stickiness near islands in phase space [11]. Zaslavsky [11] showed how these processes are described by fractional kinetic equations.

It is natural to ask if such low dimensional deterministic systems exhibit aging behaviors? And if so, how should we model aging in such systems? Here I start to answer these questions. I investigate dynamics generated by iterated maps which exhibit rich aging behavior and demonstrate how aging can be used to probe dynamics in deterministic systems. Previously, Radons [12] pointed out that *space disordered* dynamical maps may exhibit aging behavior, in analogy with random walks in random environments. As far as I know, there is no previous work relating aging to dynamics in low dimensional nonlinear and nonrandom systems.

Probably the simplest theoretical tool which generates normal and anomalous diffusion are one-dimensional maps

$$x_{t+1} = x_t + F(x_t) \tag{2}$$

with the following symmetry properties of $F(x)$: (i) $F(x)$ is periodic with a periodicity interval set to 1, $F(x) = F(x + N)$, where N is an integer. (ii) $F(x)$ has inversion antisymmetry, namely, $F(x) = -F(-x)$. The study of these maps was motivated by the assumption that they capture essential features of a driven damped motion in a periodic potential [13]. Geisel and Thomae [9] considered a rather wide family of such maps which behave as

$$F(x) = ax^z \quad \text{for } x \to +0, \tag{3}$$

where $z > 1$. Variations of these maps have been investigated by taking into account time dependent noise [14], quenched disorder [15], and additional uniform bias which breaks the symmetry of the map [16]. I use the map

$$F(x) = (2x)^z, \qquad 0 \leq x \leq \tfrac{1}{2}, \tag{4}$$

which together with the symmetry properties of the map define the mapping for all x. In Fig. 1 I show the map for three unit cells.

To investigate aging, e.g., numerically, I choose an initial condition x_{-t_a} which is chosen randomly and uniformly in the interval $-1/2 < x_{-t_a} < 1/2$. The quantity

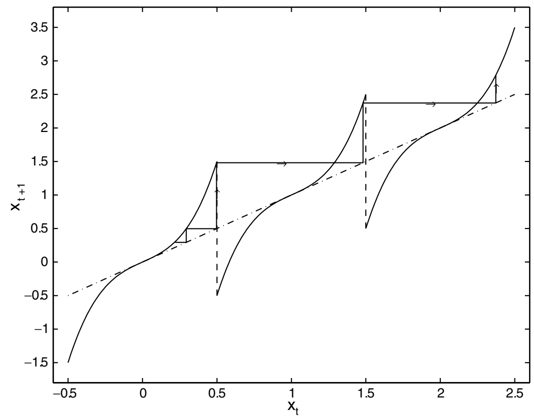

FIG. 1. The map $x_{t+1} = x_t + F(x_t)$, defined by Eq. (4) with $z = 3$. The linear dash-dotted curve is $x_{t+1} = x_t$. The unstable fixed points are on $x_t = 0, 1, 2$.

of interest is the displacement in the interval $(0, t)$, $x = x_t - x_0$, which is obtained using the map Eq. (2). Previous work [9,17] considered the nonaging regime, namely, $t_a = 0$.

In an ongoing process a walker following the iteration rules may get stuck close to the vicinity of unstable fixed points of the map (see Fig. 1). It has been shown, both analytically and numerically, that probability density function (PDF) of escape times of trajectories from the vicinity of the fixed points decays like a power law [9]. To see this, one considers the dynamics in half a unit cell, say, $0 < x < 1/2$. Assume that at time $t = 0$ the particle is on x^* residing in the vicinity of the fixed point $x = 0$. Close to the fixed point we may approximate the map Eq. (2) with the differential equation $dx/dt = F(x)$. Hence the escape time from x^* to a boundary on b ($x^* < b < 1/2$) is $t \simeq \int_{x^*}^{b} [F(x)]^{-1} dx$ using Eq. (3)

$$t \simeq \frac{1}{a}\left[\frac{x^{*-z+1}}{z-1} - \frac{b^{-z+1}}{z-1}\right]. \tag{5}$$

The PDF of escape times $\psi(t)$ is related to the unknown PDF of injection points $\eta(x^*)$, through the chain rule $\psi(t) = \eta(x^*)|dx^*/dt|$. Expanding $\eta(x^*)$ around the unstable fixed point $x^* = 0$ one finds that for large escape times [9]

$$\psi(t) \sim \frac{A}{\Gamma(-\alpha)} t^{-1-\alpha}, \qquad \alpha = \frac{1}{(z-1)}, \tag{6}$$

where A depends on the PDF of injection points. When $z > 2$ corresponding to $\alpha < 1$ two characteristic behaviors are observed. The first is that the average escape time diverges. The second, based on the work of Ignaccolo *et al.* [18], is that the invariant time independent density is never reached [19] (e.g., starting from a uniform distribution) and that the process is nonstationary. The aging I find below is related to these behaviors.

To consider stochastic properties of the aging dynamics I now investigate ACTRW [3], deriving an explicit expression for the asymptotic behavior of the Green function. Specifically consider a one-dimensional nearest neighbor lattice random walk, where lattice points correspond to the cells of the iterated maps. Waiting times on each lattice point are assumed to be described by $\psi(t)$. Note that after each jumping event it is assumed that the process is renewed, namely, we neglect correlation between motions in neighboring cells. This assumption will be justified later using numerical simulations. As mentioned, the start of the ACTRW process is at $t = -t_a$, and our goal is to find the ACTRW Green function $P(x, t_a, t)$.

In ACTRW we must introduce the distribution of the first waiting time t_1: the time elapsing between the start of observation at $t = 0$ and the first jump event in the interval $(0, t)$. Let $h_{t_a}(t_1)$ be the PDF of t_1. Let $h_s(u)$ be the double Laplace transform of $h_{t_a}(t_1)$

$$h_s(u) = \int_0^\infty dt_1 \int_0^\infty dt_a h_{t_a}(t_1) e^{-t_a s - t_1 u}. \tag{7}$$

Recently, when Gordeche and Luck [20] investigated the theory of renewal processes they showed that

$$h_s(u) = \frac{1}{1-\psi(s)} \frac{\psi(s) - \psi(u)}{u - s}, \tag{8}$$

when $z > 2$ corresponding to $\alpha < 1$ in Eq. (6)

$$h_{t_a}(t_1) \sim \frac{\sin(\pi\alpha)}{\pi} \frac{t_a^\alpha}{t_1^\alpha (t_1 + t_a)}, \tag{9}$$

which is valid in the long aging time limit. Note that Eq. (9) is independent of the exact form of $\psi(t)$, besides the exponent α. When $\alpha \to 1$ the mass of the PDF $h_{t_a}(t_1)$ is concentrated in the vicinity of $t_1 \to 0$, as expected from a "normal process." In what follows I will also use the double Laplace transform of Eq. (9)

$$h_s(u) \sim \frac{u^\alpha - s^\alpha}{s^\alpha (u - s)}. \tag{10}$$

I have checked numerically the predictions of Eq. (9) for $z = 3$, analyzing trajectories generated by the map Eq. (4) with three different aging times. In Fig. 2 I show the probability of making at least one step in the interval $(0, t)$: $\int_0^t h_{t_a}(t) dt \equiv 1 - p_0(t_a, t)$, where $p_0(t_a, t)$ is the probability of making no steps. The results show a good agreement between numerical results and the theoretical prediction Eq. (9) without fitting. Figure 2 clearly demonstrates that as the aging time becomes larger the time for the first jumping event, from one cell to its neighbor, becomes larger in the statistical sense. Thus the aging behavior is clearly related to the slow escape times from the vicinity of fixed points (when $z > 2$).

We now investigate the ACTRW Green function. Let $p_n(t_a, t)$ be the probability of making n steps in the time interval $(0, t)$. Let $P(k, s, u)$ be the double Laplace-Fourier transform ($x \to k$, $t_a \to s$, $t \to u$) of $P(x, t_a, t)$, then

$$P(k, s, u) = \sum_{n=0}^{\infty} p_n(s, u) \cos^n(k), \tag{11}$$

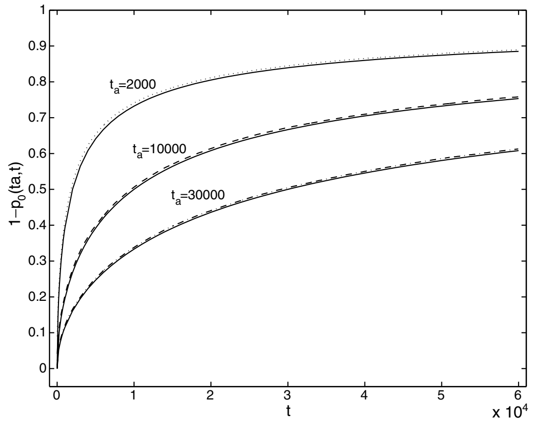

FIG. 2. The probability of making at least one step in a time interval $(0, t)$ for different aging times specified in the figure. The solid curve is the theoretical prediction Eq. (9); the dotted, dashed, and dot-dashed curves are obtained from a numerical solution of the map with $z = 3$.

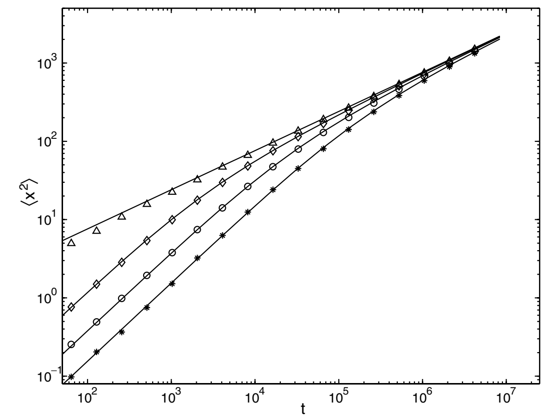

FIG. 3. The mean square displacement versus the forward time t for different aging times t_a; $t_a = 0$, triangle; $t_a = 1000$, diamond; $t_a = 10\,000$, circle; and $t_a = 60\,000$, star. The solid curves are the theoretical prediction Eq. (14). Here $z = 3$.

where $p_n(s, u)$ is the double Laplace transform of $p_n(t_a, t)$. In Eq. (11) $\cos^n(k)$ is the characteristic function of a random walk with exactly n steps. Using the convolution theorem of Laplace transform

$$p_n(s, u) = \begin{cases} \frac{1 - s h_s(u)}{su}, & n = 0, \\ h_s(u)\psi^{n-1}(u)\frac{1-\psi(u)}{u}, & n \geq 1. \end{cases} \tag{12}$$

Hence inserting Eq. (12) in Eq. (11), using Eq. (8), and summing we find

$$P(k, s, u) = \frac{1}{su} + \frac{[\psi(u) - \psi(s)][1 - \cos(k)]}{u(u - s)[1 - \psi(s)][1 - \psi(u)\cos(k)]}. \tag{13}$$

Equation (13) is a generalization of the well known Montroll-Weiss equation describing the nonequilibrium CTRW process [11,21]. Note that only if the underlying process is a Poisson process, the Green function $P(x, t_a, t)$ is independent of t_a.

Before considering the behavior of the Green function $P(x, t_a, t)$ let me consider the second moment. By differentiating Eq. (13) with respect to k twice, setting $k = 0$, and using the Tauberian theorem, I obtain the mean square displacement of the random walk for $t, t_a \gg A^{1/\alpha}$

$$\langle x^2(t_a, t)\rangle \sim \frac{1}{A\Gamma(1 + \alpha)}[(t + t_a)^\alpha - t_a^\alpha]. \tag{14}$$

For times $t \gg t_a$ I recover the standard CTRW behavior, $\langle x^2(t_a, t)\rangle \propto t^\alpha$ [21]. For $t \ll t_a$ I find $\langle x^2(t, t_a)\rangle \propto t/t_a^{1-\alpha}$; hence as t_a becomes larger, the diffusion in this regime is slowed down.

This ACTRW behavior is shown in Fig. 3 for the iterated map. Good agreement between ACTRW and the numerical simulations is found. All the curves in Fig. 3 converge into $\langle x^2(t_a, t)\rangle \propto t^\alpha$ for long forward times t, while at shorter times the diffusion is clearly slowed down as the aging time is increased.

To investigate properties of the Green function $P(x, t_a, t)$ in the limit of long t and t_a I consider the continuum approximation of Eq. (13). Inserting the large wavelength expansion $\cos(k) \sim 1 - k^2/2$ and low frequency expansion $\psi(u) \sim 1 - Au^\alpha$ in Eq. (13)

$$P(k, s, u) \sim \frac{s^\alpha u - s u^\alpha}{s^{\alpha+1}u(u - s)} + \frac{(u^\alpha - s^\alpha)}{s^\alpha(u - s)}\frac{Au^{\alpha-1}}{Au^\alpha + k^2/2}. \tag{15}$$

Inverting to the double time (t, t_a)–real space x domain one finds that the Green function is a sum of two terms:

$$P(x, t_a, t) \sim p_0(t_a, t)\delta(x) + \frac{\sin(\pi\alpha)}{\pi}\frac{1}{t_a(\frac{t}{t_a})^\alpha(1 + \frac{t}{t_a})} \otimes \frac{t|x|^{-(1+2/\alpha)}}{\alpha(2A)^{1/\alpha}} l_{\alpha/2}\left(\frac{t|x|^{-(2/\alpha)}}{(2A)^{1/\alpha}}\right), \tag{16}$$

where in this limit

$$p_0(t_a, t) \sim \frac{\sin(\pi\alpha)}{\pi}\int_{t/t_a}^{\infty}\frac{dx}{x^\alpha(1 + x)}. \tag{17}$$

The first term on the right hand side of Eq. (16) is a singular term. It corresponds to a random walk which did not make a jump in the time interval $(0, t)$. The symbol $\otimes$ in the second term in Eq. (16) is the Laplace convolution operator with respect to the forward time t, while $l_{\alpha/2}(t)$ is the one sided Lévy stable PDF, whose Laplace pair is $\exp(-u^{\alpha/2})$. In the limit $\alpha \to 1$ we get a Gaussian Green function which is independent of t_a [proof: set $\alpha = 1$ in Eq. (15)].

The behavior of the Green function Eq. (16) is shown in Fig. 4. A good agreement between simulations and the ACTRW Green function is obtained. Not shown is the singular behavior on the origin [i.e., the $\delta(x)$ term in Eq. (16)]. The behavior of this singular term is displayed in Fig. 2.

We now consider an aging diffusion equation which describes the dynamics of the iterated maps and more

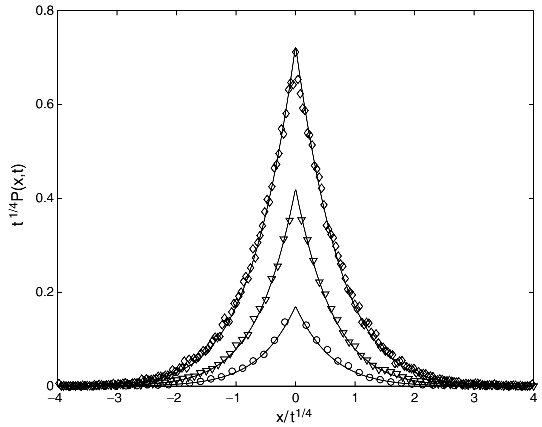

FIG. 4. The Green function obtained from simulation of the deterministic map in *scaling form* for (i) $t_a = 10^4, t = 10^3$, circles; (ii) $t_a = 10^4, t = 10^4$, triangles; and (iii) $t_a = 0$, $t = 4 * 10^5$, diamonds. The curves are our theoretical results which are in good agreement with the simulations with $z = 3$.

generally any ACTRW process. The equation is of fractional order, and it can be used to solve aging diffusion problems with different types of boundary and initial conditions. The equation is obtained from the asymptotic long time t and long aging time t_a solution of the ACTRW Eq. (16). The aging diffusion equation describes the nonsingular part of the Green function, and it reads [22]

$$A\frac{\delta^\alpha P(x, t_a, t)}{\delta t^\alpha} = \frac{1}{2}\nabla^2 P(x, t_a, t) + h_{t_a}(t) \otimes \frac{A t^{-\alpha}}{\Gamma(1-\alpha)}\delta(x), \tag{18}$$

where $\delta^\alpha/\delta t^\alpha$ is the Riemann-Liouville fractional derivative [11,21], and $h_{t_a}(t)$ is given in Eq. (9). The singular part of the Green function is specified in Eq. (16), and since the singular part of the Green function describes random walks where the number of steps is zero, boundary conditions do not alter its behavior. Equation (18) shows how to include aging dynamics into the fractional kinetic framework [11,21,23]. The second term on the right hand side of the aging diffusion Eq. (18) is a source term; we see that the age of the processes enters only through this term reflecting the fact that only statistics of the first jump event are sensitive to the age of the process. Note that Stariolo [24] used a nonlinear diffusion equation approach to model aging dynamics.

I have focused on the case $\alpha < 1$, but what happens when $\alpha > 1$? If $\alpha > 1$ (or $z < 2$), corresponding to a situation where the mean waiting time $\langle t\rangle = \int_0^\infty t\psi(t)dt$ is finite, one finds a well known behavior $\lim_{t_a\to\infty} h_{t_a}(t_1) = \int_0^{t_1}\psi(t)dt/\langle t\rangle$ [to see this insert the small s expansion $\psi(s) = 1 - s\langle t\rangle + ...$ in Eq. (8)]. We see that in this case $h_{t_a}(t_1)$ does not depend on t_a; hence for this *asymptotic* limit the map Eq. (2) and (3) will not exhibit aging behaviors. This does not imply that interesting aging behaviors cannot be observed also for $z < 2$ when t_a is finite; however, aging in this case will be observed only within a certain time window (see [25] for related work).

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