Detecting Subthreshold Events in Noisy Data by Symbolic Dynamics

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We show that a symmetric threshold crossing detector can be described by a symbolic dynamics of a static three-symbol encoding which is highly efficient to detect subthreshold events in noisy nonstationary data. After computing instantaneous word statistics and running cylinder entropies, we introduce a mean-field transformation of the three-symbol dynamics considered as a Potts-spin lattice onto a distribution of two symbols. This transformed word statistics enables one to derive an estimator of the signal-to-noise ratio (SNR). Subthreshold events are then proven by a prominent peak of the SNR estimator as a function of the noise intensity.

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The phenomenon of stochastic resonance (SR) occurs in nonlinear dynamical systems such as bistable systems [1,2] or threshold devices [3–5] when a weak input signal is enhanced by the presence of a certain level of noise (cf. [6,7] and references therein). SR has been observed in many laboratory and natural systems, among others in the earth system [1,8] and in neural systems, e.g., in ion channels [9], in models of action potentials [10], in neurons [11], in animal behavior [12], and eventually in human perception using electroencephalography [13] and event-related brain potentials (ERP) [14] (see also [15]).

In order to demonstrate SR, several measures are used. On the one hand, these are linear spectral measures, such as the power spectrum or the signal-to-noise ratio (SNR) [2]. On the other hand, residence and escape time distributions (interspike interval histograms) are used [16]. These measures have the disadvantage that they are well-defined only for SR systems with periodic forcing [10,17]. For the phenomenon of aperiodic SR the cross-correlation coefficients between the input and output of a SR device were suggested [10].

Symbolic dynamics is a natural way to describe data which appear as sequences of discrete states, such as in bistable systems, ion channels, or neurons. This approach is based on a coarse-graining of the dynamics; i.e., the time series are transformed into symbolic sequences by using very few symbols. This way one loses some amount of detailed information, whereas some of the invariant, robust properties of the dynamics are kept [18]. There is a bunch of measures of complexity which characterize such symbolic strings [19], and some of them have been applied to data from SR devices such as stochastic bistable dynamical systems and to a Schmitt trigger [20,21]. Dynamical entropies for proving SR have the same drawbacks as spectral quantities: they apply only for systems that are periodically driven. Furthermore, in order to compute mutual information, Kullback information, or cross-correlation measures, one needs both the system's input and output signals. But in natural systems often only the output time series are available. Experimentally, SR becomes manifest as a maximum of the SNR (or of the mutual information) depending on the noise amplitude. Recently, we have suggested a SNR estimator based on time averaged cylinder entropies obtained from a symbolic dynamics which can be computed for any periodic or broadband signal disturbed by additive noise. In case of periodic signals we have proved the asymptotic equivalence of this estimator with the linear SNR obtained by spectral analysis [22].

In this Letter we demonstrate that symbolic dynamics essentially exhibits SR when applied to noisy subthreshold signals. We make clear that the encoding itself serves as an amplifier of the signal. Our approach requires only a weak signal superimposed with noise; we do not need any cross-correlation or mutual information measures between signals. A further important advantage of our approach is that it can be applied to nonstationary data which typically occur, e.g., in physiological measurements, such as electroencephalogram or electrocardiogram. In order to achieve this we introduce a three-symbol encoding of a noisy signal. We calculate the SNR estimator from the instantaneous cylinder entropies [22,23] after performing a mean-field transform of the word statistics. This transformation maps a distribution of three symbols onto one of two symbols and acts as a very effective filter that should be appropriate for analyzing, e.g., neurophysiological data such as patch clamp currents of ion channels or subthreshold ERPs [14,17].

In their seminal work on SR in threshold systems Moss *et al.* discussed a periodic signal $A\sin(\omega t)$ superimposed with Gaussian colored noise [3]. We shall extend this to nonstationary data here which are the typical cases of neurophysiological data. We start by using a nonstationary test signal. Let

$$x(t) = J_0(t) + \xi(t), \quad (1)$$

where $J_0(t)$ is the zero-order Bessel function of the first kind that might be considered as a model ERP [cf.

Fig. 1(a)], while $\xi(t)$ is Gaussian noise with variance σ^2 [4]. The first step of our approach is to apply a static three-symbol encoding of this noisy signal

$$s(t) = \begin{cases} 0: & x(t) < -\theta, \\ 1: & |x(t)| \leq \theta, \\ 2: & x(t) > \theta, \end{cases} \tag{2}$$

where $\pm\theta$ are the encoding thresholds [19]; this coarse-graining is sketched in Fig. 1(b). To get a symbolic dynamics, one also needs a discretization of time which is obtained by sampling the measurement data. Since we shall deal only with instantaneous measures of complexity here, we allow for continuous time [24]. SR in such a threshold system was treated by Gammaitoni [5]. Classically, Eq. (2) defines a symmetric level-crossing detector [25], where a positive going delta spike at time t_k is emitted when $x(t_k)$ crosses the threshold θ from below, while a negative going delta spike at time t_k is emitted when $x(t_k)$ crosses the threshold $-\theta$ from above.

The symbolic dynamics defined by Eq. (2) can be interpreted as follows: A realization of the discrete-valued signal $s(t)$ is a sequence of the symbols "0," "1," and "2." Correspondingly, an ensemble of N realizations can be considered as a matrix $E = (s_{i;t})_{i \leq N; t \leq L}$ of these symbols whose rows of length L are symbolically encoded realizations of the process (1). A *cylinder set* of length l is a set of rows drawn from the matrix E which have a common building block of l symbols at a certain instance of time t (for an instructive example cf. [23]). For simplicity, we shall deal only with cylinders of length $l = 1$. Let $n_0(t), n_1(t), n_2(t)$ be the numbers of occurrence of the symbols 0, 1, and 2 in the tth column, respectively. The probability measures of the symbols are then estimated by the relative frequencies $p_k(t) = n_k(t)/N$. These probabilities constitute the *instantaneous word statistics* that can be assessed by the Shannon entropy which we refer to as cylinder entropy [22,23]. The probability to observe the symbols 0 or 2 at time t can be computed analytically yielding $p_{0,2}(t) = 0.5\,\mathrm{erfc}\{[\theta \pm J_0(t)]/(\sqrt{2}\sigma)\}$. Eventually, we obtain the probability of the symbol 1 as $p_1(t) = 1 - p_0(t) - p_2(t)$. From these probabilities the (Shannon) cylinder entropy is obtained as $H(t) = -\sum_{i=1}^{3} p_i(t)\,\mathrm{lg}\,p_i(t)$. Next, we compute the time average of the entropy over a window width Δt around a time t,

$$G = \frac{1}{\Delta t}\int_{t-\Delta t/2}^{t+\Delta t/2} H(t')dt'. \tag{3}$$

G measures the average disorder of the symbol distribution in the time interval $[t - \Delta t/2, t + \Delta t/2]$. It has high values when the noise causes many threshold crossings in the vicinity of the maxima or minima of the test function, i.e., aperiodic SR [10].

Recently, we have defined an estimator of the SNR based on averaged cylinder entropies (3) [22]. For a two-symbol encoding of periodic signals and $l = 1$ the quantity

$$S = 0.5883 \times \left(\frac{1}{G} - 1\right) \tag{4}$$

has been proven to be asymptotically equivalent to the linear SNR. Applying Eq. (4) to the symbolic dynamics (2) in the high entropy regime would yield a minimum of the estimator S. To obtain a maximum of S instead and also to filter the symbolic time series by suppressing the nonresonant regimes of the dynamics, we additionally introduce a transform of the word statistics that is inspired by mean-field theory.

Let us consider again the matrix $E = (s_{i;t})_{i \leq N; t \leq L}$ of a finite ensemble of N realizations of the symbolically encoded process (1) and (2). For a while, we shall view this matrix as a $(1 + 1)$-dimensional lattice of three-state Potts spins [26]. The realizations of the symbolic dynamics constitute a (virtual) space dimension of N lattice sites. Next, we introduce the (spatial) magnetizations $M_1(t) = p_1(t) - p_0(t)$, $M_2(t) = p_2(t) - p_0(t)$ of the spin lattice [27]. These quantities act as mean fields at the spatial lattice dimension by defining the following spin flip transformation:

$$p_0'(t) = \begin{cases} p_0(t): & M_2(t) \geq M_1(t) > 0, \\ p_0(t) + p_1(t): & M_2(t) \leq M_1(t) < 0, \\ p_0(t) + p_1(t)/2: & \text{else}, \end{cases} \tag{5}$$
$$p_1'(t) = 1 - p_0'(t),$$

which transforms the three-symbol distribution into a

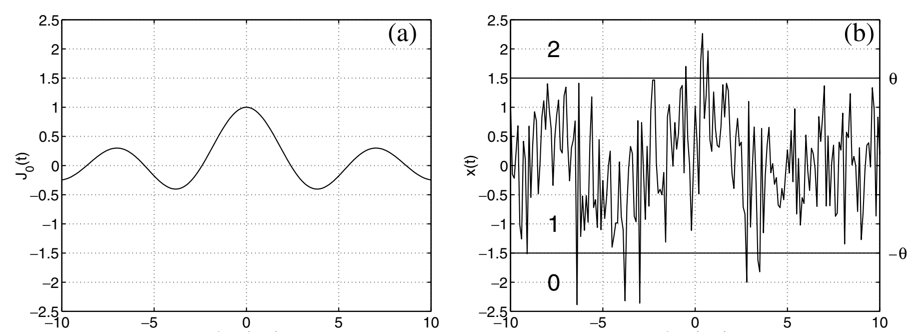

FIG. 1. The zero-order Bessel function of the first kind as an ERP-like nonstationary test signal (a), and the three-symbol encoding technique of a realization of the stochastic process Eq. (1) where $\sigma^2 = 0.64$.

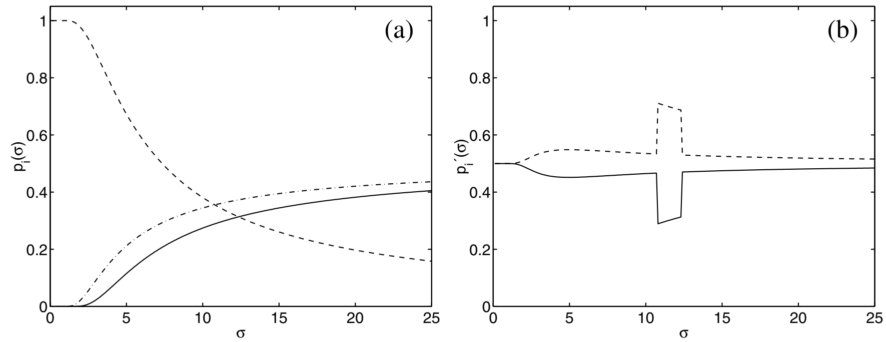

FIG. 2. Symbol statistics of the same symbolic dynamics at the global maximum of the Bessel function at $t = 0$ depending on the noise intensity for the encoding threshold $\theta = 5.0$. (a) Word statistics of the three-symbol encoding. Solid: $p_0(t)$; dashed: $p_1(t)$; dash-dotted: $p_2(t)$. (b) The mean-field transformed word statistics [due to Eq. (5)]. Solid: $p_0'(t)$, dashed: $p_1'(t)$.

distribution of two symbols depending on the cylinder measures $p_0(t)$, $p_1(t)$, and $p_2(t)$.

The symbols 0 and 2 at the temporal lattice site t should remain unchanged by this transform. But all 1's will be substituted either by 0's or by 2's depending on the magnetizations at time t, respectively. This substitution is defined as follows. We regard the resonance maximum of the averaged cylinder entropy $G(\sigma)$ at σ^* computed for a time window around the maximum of the test signal at $t = 0$. At this time the symbolic dynamics of the threshold crossing device (2) provides probably many 2's and less 0's, i.e., $p_2(t) \geq p_1(t) \geq p_0(t)$ or $M_2(t) \geq M_1(t) \geq 0$ [Fig. 2(a)]. In this case all 1's will be flipped into 2's. Since there are only two symbols 0 and 2 after applying the transform, we relabel the symbol 2 to 1. Now, the total number of 1's is given by $n_1' = n_2 + n_1$, while the frequency $n_0' = n_0$ remains unchanged. Hence, the relative frequencies of 0's and 1's are given by $p_1' = p_2 + p_1$ and $p_0' = p_0$. The latter is just obtained by the first row of Eq. (5) [cf. Fig. 2(b)]. Correspondingly, row two applies at the local minima of the test signal when all the 1's are flipped into 0's. The third row of Eq. (5) is important for the nonresonant cases of the symbolic dynamics. First, it applies to the case $\sigma < \sigma^*$ where the intermediate symbol 1 prevails. Here one-half of the 1's will be randomly substituted by 0's and the other half by 2's (relabeling 2 into 1 afterwards). Second, for $\sigma > \sigma^*$ the symbol 1 dies out but the transform acts exactly in the same way as for $\sigma < \sigma^*$.

After applying Eq. (5), the distribution of three symbols at the resonance is mapped onto a highly degenerated distribution of two symbols with low entropy and therefore a high SNR. On the other hand, the small noise regime where the symbol 1 predominates and the high noise regime where the symbol 1 gradually vanishes are mapped onto nearly uniform distributions of two symbols with high entropy and hence with a low SNR. Additionally, the transform acts as an effective noise filter. Figure 3 exhibits aperiodic SR as a distinct maximum of the SNR estimator S dependent on the noise level σ. Figure 3(a) displays the SNR estimator $S(\sigma)$ computed by Eqs. (3) and (4) at $t = 0$ using $\Delta t = 1$ after performing the transform (5) for three different encoding thresholds $\theta = 4.5, 5.0, 5.5$. The function $S(\sigma)$ is increasing for small noise intensities, it reaches a first local maximum at σ' of the order of magnitude of the threshold. We interpret this maximum of $S(\sigma)$ as the SR maximum reported by Gammaitoni [5]. In our theory his SR indicator $A(\sigma)$ is given by the distance between the dash-dotted and the solid curves of Fig. 2(a). Comparing Figs. 2(a) and 2(b) reveals that the maximum of his $A(\sigma)$ corresponds also to a maximal difference of the transformed word statistics and therefore of the SNR estimator. But it is also easy to recognize that this phenomenon could not be SR, since the prevailing symbol is 1. Aperiodic SR around the local maximum of the test signal takes place when the condition $p_2 \geq p_1 > p_0$ is fulfilled. This is the case when the function $S(\sigma)$ suddenly peaks at the resonant noise strength σ^* depending on the threshold θ. This maximum of the SNR estimator corresponds to low entropy of the mean-field transformed word statistics [Fig. 2(b)]. On the other hand, Fig. 3(b) shows

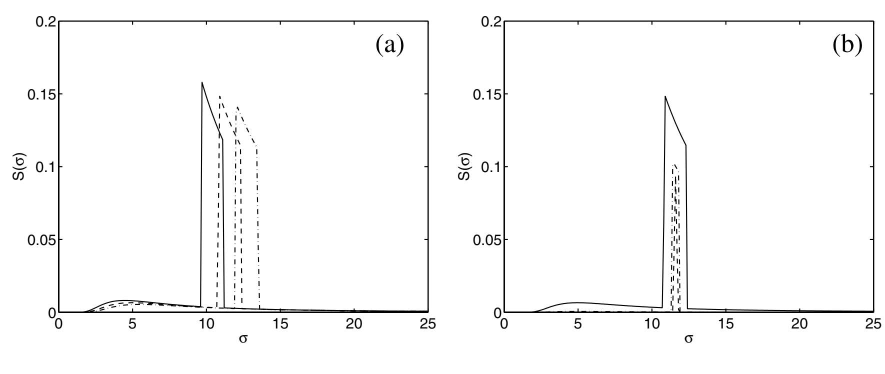

FIG. 3. SNR estimator S [Eq. (4)] of the three symbol statistics of the same system transformed to a two symbol statistics according to Eq. (5) depending on the noise intensity at $t = 0$ ($\Delta t = 1$) for (a) different encoding thresholds θ. Solid: $\theta = 4.5$; dashed: $\theta = 5.0$; dash-dotted: $\theta = 5.5$; (b) for $\theta = 5.0$ and at different times. Solid: $t = 0$; dashed: $t = 2.5$; dash-dotted: $t = 7$.

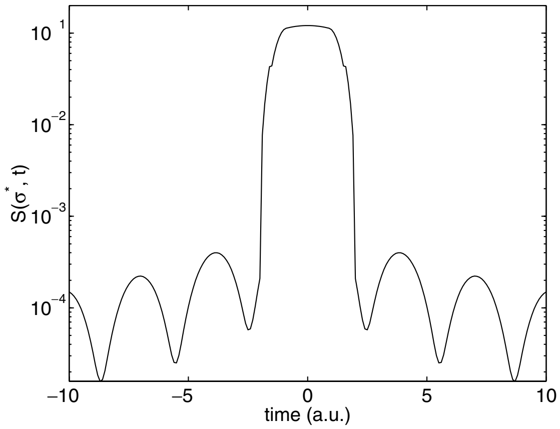

FIG. 4. Reconstructed test signal (modulo sign) using the SNR estimator $S(\sigma^*, t)$ [Eq. (4)] in a sliding window of width $\Delta t = 1$ ($\sigma^* = 12$, $\theta = 5$).

$S(\sigma)$ for $\theta = 5.0$ at the first maximum ($t = 0$), around the first zero ($t = 2.5$), and in the vicinity of the second maximum ($t = 7.0$) of the test signal $J_0(t)$. Again we have used $\Delta t = 1$. Varying Δt has the effect that the resonance peak is very sharp for smaller values and becomes blurred for larger values of Δt. The resonance peak around the zero is due to threshold crossings at the interval boundaries, but not due to the zero itself. Though there is a (trivial) maximum of $G(\sigma)$ for the three-symbol statistics in the absence of any signal, (e.g., at the zeros of the test function) this maximum is no longer present after the mean-field transform of the word statistics.

Finally, we show a reconstruction of the test signal by plotting $S(\sigma)$ at the resonant noise strength ($\sigma^* = 12$) for the threshold $\theta = 5$ against the center t of a sliding time window of width $\Delta t = 1$. This function, displayed in Fig. 4, corresponds to the absolute value of the original test signal $J_0(t)$. The logarithmic plot reveals mainly the global maximum at $t = 0$ with highly enhanced SNR. However, the local extrema of the nonstationary test signal are also recovered.

In this Letter we have reported our findings on stochastic resonance in symmetric threshold systems that are described by symbolic dynamics of three symbols. An important step is to transform the three-symbol word statistics into a distribution of only two symbols; this map has been inspired by the mean-field theory of $(1 + 1)$-dimensional lattice spin systems. From the symbolic dynamics thus obtained we have computed a SNR estimator based on time averaged running cylinder entropies. We have shown that SR is indicated by a distinct sharp peak of the SNR. We have illustrated the mean-field transform for the maxima and minima of a nonstationary noisy test signal that shares typical properties with ERP data, where local aperiodic SR takes place. In contrast to previously reported results, we do not need any cross-correlation or mutual information measure to demonstrate aperiodic SR. We have shown that the mean-field transformation of the word statistics acts as a highly effective filter. Our technique is promising for the analysis of experimental data, especially if they are nonstationary and if only the output of the system under study is available: both are typical cases in neuroscience and in the earth sciences.

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