Strong Disorder Fixed Point in Absorbing-State Phase Transitions

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The effect of quenched disorder on nonequilibrium phase transitions in the directed percolation universality class is studied by a strong disorder renormalization group approach and by density matrix renormalization group calculations. We show that for sufficiently strong disorder the critical behavior is controlled by a strong disorder fixed point and in one dimension the critical exponents are conjectured to be exact: $\beta = (3 - \sqrt{5})/2$ and $\nu_\perp = 2$. For disorder strengths outside the attractive region of this fixed point, disorder dependent critical exponents are detected. Existing numerical results in two dimensions can be interpreted within a similar scenario.

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Stochastic many particle systems with a phase transition into an absorbing state are of wide interest in physics, chemistry, and even biology [1]. Much recent work has focused on establishing a classification of possible universality classes for systems having this type of transition [2]. For models with a scalar order parameter, absence of conservation laws, and short range interactions, the critical behavior is conjectured to be that of directed percolation [3]. Well-known models with a phase transition in this universality class are the contact process [4] and the Ziff-Gulari-Barshad model of catalytic reactions [5]. When there is a conservation law present, other universality classes can appear, the best known of which is the parity conserving class [6].

In this Letter, we study the effect of spatially quenched disorder on the directed percolation universality class and show that for strong enough disorder a new universality class of absorbing state phase transitions appears.

There are two main reasons for performing such a study. First, directed percolation has been linked to several experimental situations such as catalytic reactions [5], depinning transitions [7], and the flow of granular matter [8] (for a review, see [9]). Thus far, however, the directed percolation exponents have not shown up in any of these experimental realizations. It has been suggested [9] that the presence of some form of quenched disorder is responsible for this discrepancy. Second, while several studies have been performed on the effect of spatially quenched disorder [10–15], no coherent picture of these effects has been presented. Here, we present such a scenario by relating the critical behavior of the strongly disordered contact process to that of a class of random quantum spin chains, such as the random transverse Ising model. Our results mostly concern the one-dimensional case, but we will argue that a similar picture also holds in $d = 2$.

We consider the contact process, in which each site of a lattice can either be vacant ($\varnothing$) or be occupied by one particle (A). The dynamics of the model is a continuous time Markov process in which particles at site k can disappear ($A \to \varnothing$) with a rate μ_k, while new particles can be produced on empty sites ($\varnothing \to A$), with a rate $p\lambda_k/2$, where p is the number of occupied neighbors. Here both μ_k and λ_k are independently and identically distributed random variables with distributions $\pi_0(\mu)$ and $P_0(\lambda)$, respectively. Earlier work on random versions of the contact process and of directed percolation focused mainly on dynamical aspects, by studying the properties of growth starting from a single seed particle. At criticality, a breakdown of dynamical scaling was observed [12], consistent with indications from a field theoretical study [13]. The absorbing phase was found to have properties similar to that of a Griffiths phase and shows power law behavior with nonuniversal exponents [12,14]. The static critical behavior of the model has been explored less, but exponents in $d = 2$ have been determined in [12].

In this Letter, we apply for the first time the "Hamiltonian formalism" [16,17] to the contact process with disorder. This approach has the advantage that it is ideally suited to study the asymptotic time regime, which due to the slow relaxation properties of the model [15] cannot easily be studied with simulations. Since the contact process has the property of duality [18], late and early time behavior can be related, provided suitable initial conditions are chosen. In this way, the Hamiltonian approach can also give insight in the spreading from a seed.

Consider the contact process on an open chain with L sites. The generator of the Markov process is given by the "Hamiltonian"

$$H_{CP} = \sum_{k=1}^{L} \mu_k M_k + \sum_{k=1}^{L-1} \frac{\lambda_k}{2} (n_k R_{k+1} + R_k n_{k+1}), \qquad (1)$$

in terms of the matrices

$$M = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \qquad n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

It is well known [17] that the steady state probability distribution of a stochastic process coincides with the ground state of its Hamiltonian and relaxation properties can be determined from its low lying spectrum. Although many of the Hamiltonians associated with stochastic systems are non-Hermitian, techniques developed in the study of quantum spin chains can often be successfully applied to their stochastic counterparts [17,19–21].

Here we apply a strong disorder renormalization group (RG) approach to the random contact process and the results are compared with numerical estimates calculated using exact and density matrix renormalization (DMRG) based diagonalizations. We first announce our basic results which are summarized in the schematic phase diagram in Fig. 1 as a function of the control parameter $A = [\ln\lambda]_{\rm av}$ and the strength of disorder, $R = [(\ln\lambda)^2]_{\rm av} - [\ln\lambda]^2_{\rm av}$, where, for the sake of simplicity, we put $\mu_k = 1$ (here we use $[\cdot]_{\rm av}$ to denote averaging over quenched disorder, while $\langle\cdot\rangle$ stands for the average in the steady state). In the inactive phase, for $A < A_c(R)$, the particle density $\rho \equiv \lim_{L\to\infty} \frac{1}{L}\sum_k [\langle n_k\rangle]_{\rm av}$ is zero, whereas in the active phase, for $A > A_c(R)$, $\rho > 0$. Close to the transition point, for $\Delta = (A - A_c)/A_c \downarrow 0$, the particle density is singular, $\rho \sim \Delta^\beta$, and the correlation length ξ and relaxation time τ diverge as $\xi \sim |\Delta|^{-\nu_\perp}$, $\tau \sim |\Delta|^{-\nu_\parallel}$. The anisotropy exponent is defined as $z = \nu_\parallel/\nu_\perp$. For a semi-infinite system, an exponent β_s, that describes the singular behavior of the particle density at the surface, can be introduced [22].

We found that, in one dimension, the universality class of the phase transition depends on the strength of disorder, R, and as indicated in Fig. 1 there are two different regions of critical behavior. For $R > R_c$, the critical properties of the system are controlled by a strong disorder fixed point, which is situated at $R = \infty$, and in which the critical exponents are conjecturedly exact:

$$\beta^{(\infty)} = \frac{3-\sqrt{5}}{2}, \qquad \beta_s^{(\infty)} = 1, \qquad \nu_\perp^{(\infty)} = 2. \tag{2}$$

This fixed point is characterized by strongly anisotropic scaling.

$$\ln\tau \sim \xi^\psi, \qquad \psi = \tfrac{1}{2}. \tag{3}$$

Thus, the anisotropy exponent, $z^{(\infty)}$, is formally infinity.

For $R \le R_c$, as shown in Fig. 1, there is a line of disorder dependent random fixed points (with finite z), in which the critical exponents vary continuously with R. This behavior is consistent with an extension of the Harris criterion for stochastic systems, which predicts a crossover exponent in the nonrandom system fixed point as $y_{\rm dis}(0) = 2/\nu_\perp(0) - 1 > 0$ [1,10]. Indeed, applying a real space RG method [21], we have calculated $y_{\rm dis}(0)$ in good agreement with the Harris criterion.

In the strong disorder RG, which was introduced [23] for random antiferromagnetic spin chains and which was later also applied to other one- and higher-dimensional systems [24–26], one puts the strength of the couplings and fields in the Hamiltonian (i.e., the rates in the stochastic problem) in descending order. The strongest one, denoted by Ω, sets the energy scale in the system and is decimated out, and the neighboring rates are replaced by (generally) weaker ones, obtained by a perturbation calculation.

In the random contact process, if the fastest rate is $\lambda_i = \Omega$, then the sites i and $i+1$ will either be both occupied or both empty almost always, and can thus be interpreted as an effective two-site cell state. The effective decay rate, μ', to go from AA to $\varnothing\varnothing$ can then be determined by applying perturbation theory to the Hamiltonian (1). At the same time, the effective moment of the cell state, n', is given by $n' = n_i + n_{i+1}$, where in the starting situation $n_k = 1$, $\forall k$. Conversely, if the fastest rate is $\mu_i = \Omega$, site i will almost always be empty, and in a perturbative treatment can be eliminated from the system. This allows the determination of an effective coupling λ' between the occupied site $i-1$ and the empty site $i+1$. In terms of $\tilde\lambda = \lambda/\sqrt{8}$, we obtain the decimation equations ($\kappa = \sqrt{2}$):

$$\mu' = \kappa\frac{\mu_i\mu_{i+1}}{\tilde\lambda_i}, \qquad n' = n_i + n_{i+1},$$
$$\tilde\lambda' = \kappa\frac{\tilde\lambda_{i-1}\tilde\lambda_i}{\mu_i}. \tag{4}$$

This decimation procedure is repeated and for strong enough disorder, when the probability of generating a new rate larger than Ω is negligible, the energy scale is

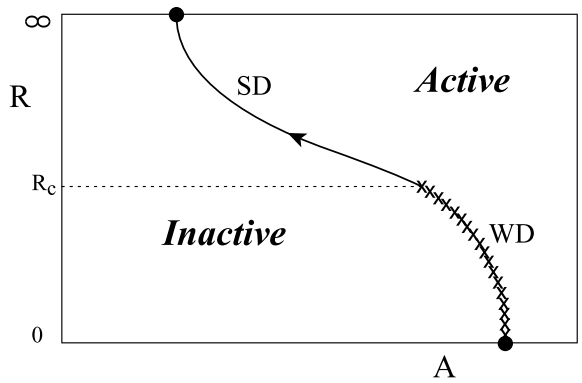

FIG. 1. Schematic phase diagram of the random contact process as a function of A and R (see text). The inactive and active phases are separated by a phase transition line, along which the critical exponents are R dependent for weaker disorder (WD, $R \le R_c$), or are determined by a strong disorder fixed point at $R = \infty$, for strong disorder (SD, $R > R_c$).

continuously lowered and at the same time the probability distributions, $\pi(\mu, \Omega)$ and $P(\lambda, \Omega)$, approach their fixed point forms. This RG procedure is easy to implement numerically. However, in the 1D case the fixed point distributions can be calculated analytically, using the results of a similar analysis for 1D random quantum spin chains [24–26]. Here we mention that according to Eq. (4) the renormalized log-μ rates after large L decimation steps are expressed as the sum of $L + 1$ original random log-μ rates minus the sum of L original random log-$\tilde{\lambda}$ rates (plus a negligible constant proportional to $\ln\kappa$). Consequently, the fixed point, where the distributions $\pi(\mu, \Omega)$ and $P(\tilde{\lambda}, \Omega)$ are identical, is located at $[\ln\mu]_{\rm av} = [\ln\tilde{\lambda}]_{\rm av}$. Furthermore, at the fixed point the central limit theorem implies for the log-energy scale $-\ln\Omega \sim L^{1/2}$, which is equivalent to (3). As a consequence, the probability distributions of the rates are broadened without limit during renormalization. Thus, the ratio of typical neighboring rates goes to either zero or infinity. Therefore, in the strong disorder fixed point the decimation equations in Eq. (4) become asymptotically exact. Thus, the critical exponents in this fixed point [see (2)], which can be deduced from the analysis of the RG equations along similar lines as for the random transverse Ising model [24], are presumably exact also.

In the following, we check these predictions by numerical calculations on lattices with L up to 32 with the DMRG method. For the disorder, we used a bimodal distribution with $\mu_k = 1$ and $P(\lambda) = [\delta(\lambda - \lambda_+) + \delta(\lambda - \lambda_-)]/2$, where $\lambda_\pm = \exp(A \pm \sqrt{R})$, so that A and R are the parameters used in Fig. 1. For smaller sizes, $L \leq 14$, we performed exact disorder averages, whereas for larger L we considered at least 10 000 samples for each size.

In the actual calculation, we set $\mu_L = 0$ and determine the average particle density profile, $\rho_k = [\langle n_k \rangle]_{\rm av}$, so that the order parameter in the bulk (at the surface) is defined as $\rho = \rho_{L/2}$ ($\rho_s = \rho_1$). Close to the critical point, the densities obey scaling laws, such as $\rho(L, \Delta) = L^{-x}\tilde{\rho}(\Delta L^{1/\nu_\perp})$, where $x = \beta/\nu_\perp$. A similar relation holds for ρ_s, with the surface exponent $x_s = \beta_s/\nu_\perp$. From an optimal data collapse, as illustrated in Fig. 2, the position of the transition point, A_c, and the critical exponents are calculated.

For varying disorder strength, R, the estimated critical exponents, x and x_s, are plotted in Fig. 3, where one can observe the two regions announced in Fig. 1. The crossover occurs near $R_c \approx 1.5$. For higher disorder strengths, the critical exponents seem to saturate at the values given in (2). The estimate for $\nu_\perp = 1.7 \pm 0.3$ at $R = 1.5$ is also consistent with $\nu_\perp^{(\infty)}$. The error on this exponent is larger because of stronger finite size corrections.

We note that the same type of weak-to-strong disorder crossover scenario has been observed in a class of random quantum spin chains [27]. For these models, the strong disorder RG analysis leads to decimation equations

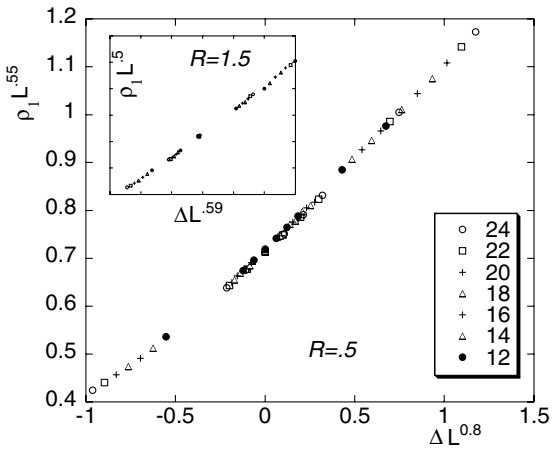

FIG. 2. Scaling plot of the surface particle density at $R = 0.5$ and at $R = 1.5$ (inset).

of the same form as Eq. (4), with also $\kappa > 1$. For these κ values, some of the generated couplings (rates) can be larger than the decimated one. For weak enough disorder, this happens so often that the RG approach does not work. For stronger, but still finite disorder, however, such types of decimation steps are rare so that during renormalization the system flows into the strong disorder fixed point [27].

We have also studied the dynamical scaling of the model by calculating the probability distribution of the gap Γ between the ground state and the first excited state of H_{CP}. In the small Γ limit, this distribution behaves as $P(\ln\Gamma) \sim \Gamma^{1/z'}$. Here z' is the disorder induced dynamical exponent, and the true dynamical exponent of the system is given by $z = \max[z', z(0)]$. In the strong disorder fixed point z' is formally infinity and the scaling behavior of the gap distribution is given by $P(\ln\Gamma) = \tilde{P}(\ln\Gamma/\sqrt{L})/\sqrt{L}$. As shown in Fig. 4, the above scaling form is indeed satisfied in the strong disorder region at $R = 1.5$.

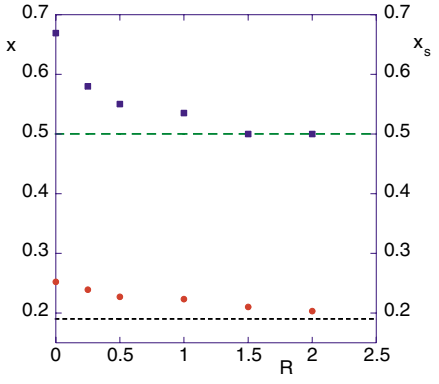

FIG. 3 (color online). Numerical estimates of the exponents $x = \beta/\nu_\perp$ (circles) and x_s (squares). The broken lines indicate the value at the strong disorder fixed point. The errors are of the same order as the size of the symbols.

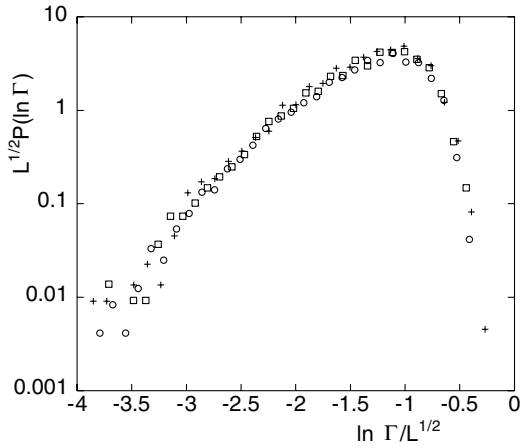

FIG. 4. Scaling of the gap to the first excited state at $R = 1.5$ and $A = A_c = 1.18$ [$L = 16$ (+), 20 (□), 24 (○)].

Based on the decimation rules in Eq. (4), which are very similar to that of the random transverse Ising model, we expect that even in higher dimensions these random stochastic models will renormalize into the strong disorder fixed point for the higher-dimensional random transverse Ising model [28]. Indeed from [28] we obtain $\nu_\perp = 1.07 \pm 0.15$ and $\beta = 1.07 \pm 0.12$, consistent with the values obtained for a contact process with sufficiently strong dilution [12]. For smaller values of the dilution, β seems to vary continuously [12], indicating that a scenario similar to that in $d = 1$ is valid. Moreover, the breakdown of dynamical scaling observed in [12] can be related to the strongly anisotropic scaling (3). Using $\psi = 0.42 \pm 0.06$ [28], it is even possible to predict [29] the values of some of the dynamic exponents introduced in [12].

In conclusion, we have found that the critical properties of absorbing states models in the directed percolation universality class in the presence of random transition rates are controlled by a strong disorder fixed point, if the disorder is sufficiently strong. The exponents associated with this transition are conjectured exact. This is the first example of an absorbing state phase transition with quenched disorder for which exact information becomes available. A more extensive paper on our results will be published elsewhere [29].

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