Euler-Lagrange Correspondence of Cellular Automaton for Traffic-Flow Models

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We propose a *Euler-Lagrange transformation* for cellular automata (CA) by developing new explicit transformation formulas. This transformation is done in the fully discrete level of variables, and corresponds to the well-known continuous version of it which appears in continuous mechanics such as fluid dynamics and plasma physics. Applying this method to the traffic problem, we have obtained the Lagrange representation of a traffic model, and also succeeded in clarifying the relation between different types of traffic models. It is shown that the Burgers CA, which is a corresponding CA of the continuous Burgers equation, plays a central role in considering this relation.

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Recently, cellular automata (CA) have been extensively used for modeling complex phenomena in various fields such as fluid dynamics, statistical physics, biology, and other complex systems [1]. Among these CA, the rule-184 CA, which is one of the elementary CA (ECA) proposed by Wolfram [2], has attracted much attention as a model of dynamics of interface [3] and traffic flow [4,5]. For the model of traffic flow, the rule-184 CA is known to represent the minimal model for movement of vehicles in one lane and show a simple phase transition from free to congested state of traffic flow [6].

On the other hand, one of the authors has proposed the so-called ultradiscrete method by which we obtain corresponding CA from difference equations [7,8]. By applying this method to various integrable equations, we have so far obtained several CA which inherit many mathematical properties of corresponding difference equations. The examples include the family of the box-and-ball system and Toda CA [9,10], which are all integrable in both level of difference and ultradiscrete equation.

In the previous paper [11], by using the ultradiscrete method, the Burgers CA(BCA) has been derived from the Burgers equation $\rho_t = 2\rho\rho_x + \rho_{xx}$ which was used by Musha *et al.* as a macroscopic traffic model [12,13]. BCA is written as

$$U_j^{t+1} = U_j^t + \min(U_{j-1}^t, L - U_j^t) - \min(U_j^t, L - U_{j+1}^t), \tag{1}$$

where U_j^t denote the number of vehicles at the site j and time t. The parameter L represents the maximum capacity of a cell and is related to the lattice interval of the spatial coordinate x [11]. BCA can be regarded as a kind of cell transmission model, and is similar to the one proposed by Daganzo [14]. Putting the restriction $L = 1$ on (1), BCA is found to be equivalent to the rule-184 CA, which is a prototype of the microscopic traffic models [4]. Since the Burgers equation is considered to be the one-dimensional Navier-Stokes equation, it is natural to, say, that (1) is the *Euler representation* of traffic flow. The above result has clarified the fact that there is a rigorous relation between a macroscopic traffic model and a microscopic one in the Euler representation via the ultradiscrete method. In the Euler description, flow is observed at a certain fixed point in space and a dependent variable represents the amplitude of a field at that point. In the case of traffic flow, the field variable is usually taken as the density of the vehicles at a certain cross section of a road and vehicles are not distinguished individually in this description.

There is the other representation called *Lagrange representation*, which originally also comes from hydrodynamics. In this representation, we observe each particle and follow the trajectory of it. Thus each vehicle is considered to be a distinguishable interacting particle, and the dependent variable in this case represents the position of each vehicle. This representation is known to be suitable for the case that the order of vehicles does not change, i.e., every car does not overtake the car in front of it [15]. So far there has been proposed several Lagrange-type traffic models, which is called car-following models, such as the optimal velocity (OV) model [16] and the intelligent driver model [13,17]. These are all classified as a microscopic model. The relation between these models and macroscopic models of the Euler type is recently discussed by Lee *et al.* using perturbation techniques [18] and by Helbing *et al.* using numerical simulation [19]. In the microscopic case, Boccara and Fukś discuss about the examples of CA which can be regarded as particle systems and analyze these using a theory of mapping [20,21]. However, the relation between Euler and Lagrange models for CA has not fully been clarified up to now.

In this Letter, we propose the Euler-Lagrange transformation by developing new explicit transformation formulas containing the max and step function, and connect the missing link in the microscopic models of traffic flow of the Euler and Lagrange form. By this transformation, we will derive the Lagrange representation of the rule-184

CA from BCA, which will correspond to a car-following model.

Let us consider the Euler representation of rule-184, i.e., BCA in the case of $L = 1$,

$$U_j^{t+1} = U_j^t + \min(U_{j-1}^t, 1 - U_j^t) - \min(U_j^t, 1 - U_{j+1}^t). \tag{2}$$

Introducing the variable S by

$$S_j^t = \sum_{k=-\infty}^{j} U_k^t, \tag{3}$$

or $U_j^t = S_j^t - S_{j-1}^t$, and rewriting (2) in S, we obtain

$$S_j^{t+1} = \max(S_{j-1}^t, S_{j+1}^t - 1). \tag{4}$$

S_j^t is the total number of the vehicles from $-\infty$ to jth site, which is assumed to be finite. Note here that if we introduce dependent variable transformation $S_j^t = F_j^t + \frac{j}{2} - \frac{t}{2}$, (4) becomes the ultradiscrete diffusion equation [11],

$$F_j^{t+1} = \max(F_{j-1}^t, F_{j+1}^t). \tag{5}$$

Here, we put

$$S_j^t = \sum_{i=0}^{N-1} H(j - x_i^t), \tag{6}$$

where $H(x)$ is the step function defined by $H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ otherwise, and N is the total number of vehicles on the road. x_i^t is the Lagrange variable that represents the position of the ith car at time t and the relation $x_0^t < x_1^t < \cdots < x_{N-1}^t$ holds. Equation (6) shows the relation between the Euler variable S and the Lagrange variable x.

Using (6) to replace S in (4) by H, we have

$$\sum_{i=0}^{N-1} H(j - x_i^{t+1}) = \max\left[\sum_{i=0}^{N-1} H(j - x_i^t - 1), \sum_{i=0}^{N-1} H(j - x_i^t + 1) - 1\right]. \tag{7}$$

Here we introduce a new identity for max function and step function,

$$H[j - \min(a, b)] + H[j - \min(c, d)] = \max[H(j-a) + H(j-c), H(j-b) + H(j-d)], \tag{8}$$

which holds for any constants $a < c$, $b < d$ (Fig. 1).

Generalizing (8), we get

$$\sum_{k=1}^{n} H[j - \min(a_k, b_k)] = \max\left[\sum_{k=1}^{n} H(j - a_k), \sum_{k=1}^{n} H(j - b_k)\right], \tag{9}$$

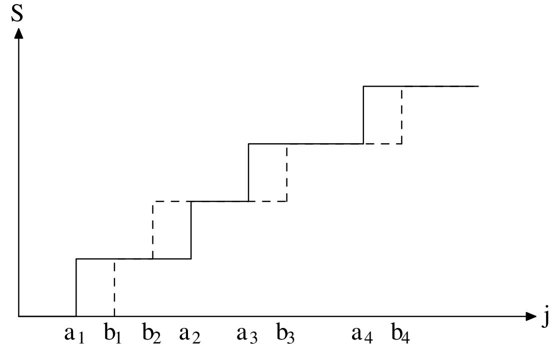

FIG. 1. Illustration of formulas (8) and (9). The solid line denotes $\sum_{k=1}^{n} H(j - a_k)$ and the dotted line denotes $\sum_{k=1}^{n} H(j - b_k)$. The max operation of the two lines corresponds to take the *envelope* of these lines.

assuming $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$ (Fig. 1). This formula expresses the commutability of max function and step function which allows us to manipulate Lagrange variables, and hence it is considered to be a fundamental formula for the Euler-Lagrange transformation.

Another important formula can be obtained by considering an equivalence of a vertical and horizontal shift shown in Fig. 2, which is expressed by

$$\max\left[\sum_i H(j - a_i^t) - m, 0\right] = \sum_i H(j - a_{i+m}^t), \tag{10}$$

where we assume that $a_j^t = \infty$ if j is larger than the number of vehicles. This formula shows that the subtraction of m in the left-hand side is the same as the shift of m in the subscript of x with the help of the operation $\max(x, 0)$ which takes only the positive part of x.

Since $x_0^t < x_1^t < \cdots < x_{N-1}^t$, (7) can be transformed into

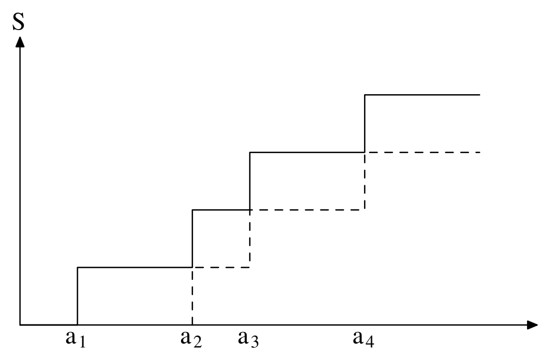

FIG. 2. Illustration of formula (10). The solid line denotes $\sum_{k=1}^{n} H(j - a_k)$ and the dotted line denotes $\max[\sum_{k=1}^{n} H(j - a_k) - 1, 0]$, which is the same as $\sum_{k=1}^{n} H(j - a_{k+1})$. The $-m$ shift to the vertical direction is the same as the horizontal shift of m.

$$\sum_i H(j - x_i^{t+1}) = \max\left[\sum_i H(j - x_i^t - 1), \sum_i H(j - x_{i+1}^t + 1)\right] = \sum_i H[j - \min(x_i^t + 1, x_{i+1}^t - 1)], \tag{11}$$

by using (9) and (10). Comparing both sides, we finally obtain

$$x_i^{t+1} = \min(x_i^t + 1, x_{i+1}^t - 1) = x_i^t + \min(1, x_{i+1}^t - x_i^t - 1), \tag{12}$$

which is the Lagrange representation of rule-184.

Adding the term $-(x_i^t - x_i^{t-1})$ to both sides in (12) gives

$$x_i^{t+1} - 2x_i^t + x_{i-1}^t = \min(1, h_i^t) - (x_i^t - x_i^{t-1}), \tag{13}$$

where $h_i^t = x_{i+1}^t - x_i^t - 1$ is the headway of the ith car, which represents the number of vacant sites between the ith and $(i+1)$th car. The Taylor expansion of (13) gives [22]

$$\frac{d^2 x_i}{dt^2} = a\left[V \min(1, h_i) - \frac{dx_i}{dt}\right], \tag{14}$$

where we have introduced the parameter a and V, which have both the order of $\sim 1/\Delta t$. This is nothing but the OV model, and $V \min(1, h_i)$ corresponds to the piecewise linear OV function which appeared in [23]. The parameter a is related to the sensitivity of drivers and is known to be proportional to the inverse of a characteristic time [16]. The other parameter V has the role of scaling the min function, i.e., the optimal velocity function. These arbitrary parameters are usually determined empirically by using observed data of traffic flow.

In the case of $L > 1$, BCA becomes multivalued CA and we need further techniques. We introduce the variable u by

$$S_j^t = \sum_{k=-\infty}^{j} u_k^t, \quad U_j^t = u_{Lj+1}^t + u_{Lj+2}^t + \cdots + u_{L(j+1)}^t = S_{L(j+1)}^t - S_{Lj}^t, \tag{15}$$

where u_j^t denotes the number of vehicles whose value is zero or one at jth site at time t (see Fig. 3).

Substituting (15) into (1), we obtain

$$S_{Lj}^{t+1} = \max[S_{L(j-1)}^t, S_{L(j+1)}^t - L]. \tag{16}$$

Replacing Lj by j, this becomes

$$S_j^{t+1} = \max(S_{j-L}^t, S_{j+L}^t - L). \tag{17}$$

Note that if we put $S_j^t = F_j^t + \frac{j}{2} - \frac{Lt}{2}$, this becomes the ultradiscrete diffusion equation,

$$F_j^{t+1} = \max(F_{j-L}^t, F_{j+L}^t). \tag{18}$$

$\longrightarrow j$

u	000	100	010	001	110	111
S	000	111	122	223	455	678
U	0	1	1	1	2	3

FIG. 3. The relation between u, S, U when $L = 3$. u and S are divided by vertical lines with the period L. U takes the multivalue of integer between 0 and L. This corresponds to the multilane model of traffic flow in a coarse-grained sense [24]. Note that all the results are not affected by the position of vertical lines [22], which is one of the remarkable properties of BCA.

By setting

$$S_j^t = \sum_{i=0}^{N-1} H(j - x_i^t), \tag{19}$$

we obtain the Lagrange form of BCA with general L by the similar procedures as

$$x_i^{t+1} = x_i^t + \min(L, x_{i+L}^t - x_i^t - L). \tag{20}$$

We should note here that the derived Eq. (20) is a special case of

$$x_i^{t+1} = x_i^t + \min(V, x_{i+S}^t - x_i^t - S), \tag{21}$$

where V and S are parameters and $V \neq S$ in general. This equation contains the Fukui-Ishibashi model [4] and the quick-to-start model [25] as special cases by putting $S = 1$ and $V = 1$, respectively, thus it is considered as a generalization of these CA models of traffic flow.

One time step evolution of the (20) is shown by Fig. 4 in the case of $L = 3$.

The relation between (1) and (20) constitutes the missing link in the microscopic traffic models.

In this Letter, we have derived the Lagrange representation of the rule-184 and its generalization from BCA. From a practical point of view, Euler-Lagrange correspondence is quite important for traffic problems. Statistical data such as traffic flow and traffic density are taken by the loop coils in the road [26], which is nothing but the Euler description. On the other hand, computer simulations are sometimes performed by the equations in Lagrange forms if we are interested in the

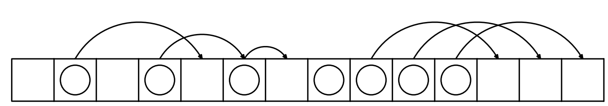

FIG. 4. The Lagrange representation of BCA. The particles move at most L sites in a step without overtaking the particle in each front. L is considered as the maximum speed as well as the number of cars that a driver can see in front [22].

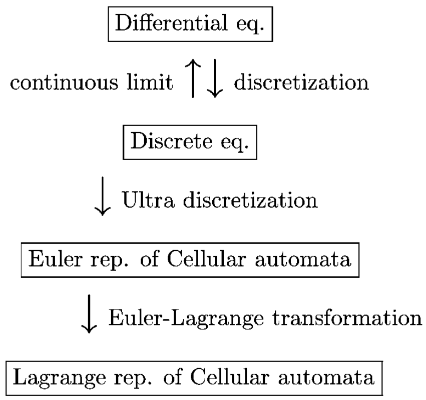

FIG. 5. The route from differential equation to the Lagrange representation of CA.

individuality of vehicles. In order to analyze observed data by computer simulation, we need to make clear the correspondence between Euler and Lagrange representation of models and data. From mathematical points of view, Euler-Lagrange correspondence is also helpful to the understanding of the properties of nonlinear equations. There are several types of equations such as differential equations, discrete equations, and ultradiscrete equation (CA), depending on the discreteness of variables. The relation between these equations has been established recently by the discovery of the ultradiscretization. The further variations of the above equations can be discussed and hidden equivalence between different types of equations will be studied by using the Euler-Lagrange transformation (Fig. 5).

In the traffic flow models, there are several other CA models, which allows both Euler and Lagrange representation, such as the slow-to-start model [27] and Nagel-Schreckenberg model [5]. As our method is applied to flow-conserved systems in Euler form, we expect the method to be applicable to these models. Now we are progressing on finding the Euler-Lagrange correspondence of these models and shall report them in successive papers. Furthermore, the Euler form of a two-dimensional generalization of BCA has been recently proposed [28]. Its transformation and applications will be also investigated by using this method in the future. We believe that establishing the general Euler-Lagrange correspondence of CA will make a new development of the ultradiscretization method and studies of CA.

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