Statistics of Lagrangian Velocities in Turbulent Flows

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We present a generalized Fokker-Planck equation for the joint position-velocity probability distribution of a single fluid particle in a turbulent flow. Based on a simple estimate, the diffusion term is related to the two-point two-time Eulerian acceleration-acceleration correlation. Dimensional analysis yields a velocity increment probability distribution with normal scaling $v \approx t^{1/2}$. However, the statistics need not be Gaussian.

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The question, how marked fluid particles behave in a turbulent flow, is one of the central issues in the theory of turbulence [1–4]. It is intimately related to the problem of turbulent diffusion and mixing. Recently, experiments [5,6] have been performed, which allow one to quantify in a statistical manner the behavior of single marker particles. The behavior of a single particle, initially starting at the position $\mathbf{y}$, has been examined in great detail by considering its path $\mathbf{X}(t, \mathbf{y})$ and velocity $\mathbf{U}(t, \mathbf{y})$. The central quantity of interest is the probability distribution function (PDF)

$$f(\mathbf{u}, \mathbf{x}, \mathbf{y}, t) = \langle \delta(\mathbf{u} - \mathbf{U}(t, \mathbf{y}))\delta(\mathbf{x} - \mathbf{X}(t, \mathbf{y}))\rangle, \quad (1)$$

where the brackets indicate a suitable average. In the experiments [6], the PDF $h(v, \tau)$ of the velocity increment $\mathbf{v}_t(\tau, \mathbf{y}) = \mathbf{U}(t + \tau, \mathbf{y}) - \mathbf{U}(t, \mathbf{y})$ has been measured for various values of the time delay τ for the case of stationary turbulence. Whereas standard scaling arguments along the lines of Kolmogorov's theory [2] suggest a simple scaling behavior of the moments of the increments $\langle v(\tau, \mathbf{y})^n\rangle \approx \tau^{n/2}$, the measured moments show multifractal scaling behavior, $\langle v(\tau, \mathbf{y})^n\rangle \approx \tau^{\zeta_n}$, where $\zeta_n = n/2 + \chi(n)$ is a nonlinear function of n. This multifractal scaling is a result of the phenomenon of intermittency [6]. Intermittency manifests itself in a non-Gaussian shape of the probability distribution of the increments: Its shape changes from a strongly non-Gaussian one at small time delays τ up to a Gaussian at large time scales (cf. Fig. 1). Recently, it has been suggested that this type of intermittency is related to long time correlations of the acceleration [7].

An early model for the PDF (1) was devised by Oboukhov [8], who assumed that the turbulent acceleration $\dot{\mathbf{U}}(t, \mathbf{y}) = \mathbf{a}(t, \mathbf{y})$ is a Gaussian, δ-correlated random force, $\langle a_i(t, \mathbf{y})a_j(t', \mathbf{y})\rangle = 2Q\delta_{ij}\delta(t - t')$, such that the PDF obeys a Fokker-Planck equation:

$$\dot{f}(\mathbf{u}, \mathbf{x}, \mathbf{y}, t) + \mathbf{u} \cdot \nabla f(\mathbf{u}, \mathbf{x}, \mathbf{y}, t) = Q\Delta_{\underline{u}} f(\mathbf{u}, \mathbf{x}, \mathbf{y}, t). \quad (2)$$

For stationary turbulence, the PDF of the velocity increment $\mathbf{v}(\tau, \mathbf{y})$ equals $h(\mathbf{v}, \tau) = \int d\mathbf{x} f(\mathbf{u}, \mathbf{x}, \mathbf{y}; \tau)$ with initial condition $h(\mathbf{v}, 0) = \delta(\mathbf{v})$. The Oboukhov model yields the scaling behavior $\langle v(\tau)^2\rangle \approx \tau$, which is theoretically supported by standard dimensional arguments along Kolmogorov's theory of 1941 [2]. However, until now, except for [9] no attempt has been made to derive (2) by an analysis of the Navier-Stokes equation. Recently, I have approached this problem by considering evolution equations for N-particle PDFs [10]. Thereby, an infinite hierarchy of evolution equations for the Lagrangian PDFs of N marker particles is formulated. The evolution equation for the PDF of N marker particles contains the PDF of $N + 1$ particles. One tries to describe the statistical behavior of N marker particles by a closure of the hierarchy, i.e., by expressing the PDF of $N + 1$ particles as a functional of the PDF of N particles. The present Letter aims at describing results for the single-particle PDF. A simple closure assumption will yield an extension of the Fokker-Planck equation (2) including a memory term. In contrast to Oboukhov's model, this generalized Fokker-Planck equation has scaling solutions with non-Gaussian

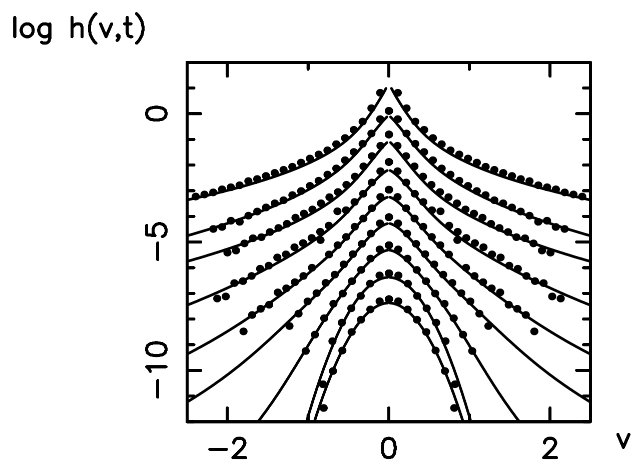

FIG. 1. A comparison of probability distributions $1/\sqrt{u(t)^2} h[u/\sqrt{u(t)^2}, t]$, Eq. (17), for various values of ν with the experimentally obtained ones determined by the Lyon group [6] [the curves are shifted (from above: $t = 0.15$, 0.3, 0.6, 1.2, 2.5, 5.0, 10.0, 20.0, and 40.0 ms) and fitted by the values $\nu = 1.71$, 1.48, 1.36, 1.26, 1.14, 1.06, 0.94, 0.70, and 0.70].

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PDFs and, therefore, can account for the recent findings [6]. Our result supports the conjecture [7] that long time correlations are at the origin of the intermittent shapes of the PDFs, since the obtained Fokker-Planck equation contains a time memory.

Let us start by establishing an evolution equation of the one particle PDF $f(\mathbf{u}, \mathbf{x}, \mathbf{y}, t)$ involving the Navier-Stokes dynamics. Using standard procedures [10], one can derive the following equation $f(\mathbf{u}, \mathbf{x}, \mathbf{y}, t)$:

$$\dot{f}(\mathbf{u}, \mathbf{x}, \mathbf{y}, t) + \mathbf{u}\cdot\nabla_{\mathbf{x}} f(\mathbf{u}, \mathbf{x}, \mathbf{y}, t) = -\nabla_{\mathbf{u}}\cdot\int d\mathbf{v}\, d\mathbf{r}\, \mathbf{A}(\mathbf{x}-\mathbf{r}, \mathbf{v}) f(\mathbf{v}, \mathbf{r}, t; \mathbf{u}, \mathbf{x}, \mathbf{y}, t). \tag{3}$$

For an infinitely extended fluid, the operator $\mathbf{A}(\mathbf{x}-\mathbf{r})$ is related to the pressure term and the dissipation term in the Navier-Stokes equation and is explicitly given by

$$A_i(\mathbf{x}-\mathbf{r}, \mathbf{v}) = -\sum_{jk} \frac{\partial^3}{\partial x_i\, \partial x_j\, \partial x_k} \frac{1}{4\pi|\mathbf{x}-\mathbf{r}|} v_j v_k + \nu\delta(\mathbf{x}-\mathbf{r})\Delta_{\mathbf{r}} v_i. \tag{4}$$

The evolution equation for the one point PDF is not closed. It involves a mixed Eulerian-Lagrangian PDF $f(\mathbf{v}, \mathbf{r}, t; \mathbf{u}, \mathbf{x}, \mathbf{y}, t' = t)$, which is defined according to

$$f(\mathbf{v}, \mathbf{r}, t; \mathbf{u}, \mathbf{x}, \mathbf{y}, t') = \langle \delta(\mathbf{v}-\mathbf{V}(\mathbf{r}, t))\, \delta(\mathbf{u}-\mathbf{U}(t', \mathbf{y})) \times \delta(\mathbf{x}-\mathbf{X}(t', \mathbf{y}))\rangle. \tag{5}$$

Here, $\mathbf{V}(\mathbf{r}, t)$ denotes the Eulerian velocity field at location $\mathbf{r}$ and time t. Let us consider the evolution equation for the probability distribution $f(\mathbf{v}, \mathbf{r}, t; \mathbf{u}, \mathbf{x}, \mathbf{y}, t')$:

$$\frac{\partial}{\partial t'} f(\mathbf{v}, \mathbf{r}, t; \mathbf{u}, \mathbf{x}, \mathbf{y}, t') + \mathbf{u}\cdot\nabla_{\mathbf{x}} f(\mathbf{v}, \mathbf{r}, t; \mathbf{u}, \mathbf{x}, \mathbf{y}, t') = -\nabla_{\mathbf{u}}\cdot\int d\mathbf{v}'\, d\mathbf{r}'\, \mathbf{A}(\mathbf{x}-\mathbf{r}', \mathbf{v}') f(\mathbf{v}', \mathbf{r}', t'; \mathbf{v}, \mathbf{r}, t; \mathbf{u}, \mathbf{x}, \mathbf{y}, t'). \tag{6}$$

The evolution equation for $f(\mathbf{v}, \mathbf{r}, t; \mathbf{u}, \mathbf{x}, \mathbf{y}, t')$ contains a three point probability distribution, where $\mathbf{v}, \mathbf{v}'$ are Eulerian fields at the points $(t, \mathbf{r})$ and $(t', \mathbf{r}')$, respectively. In principle, a hierarchy of evolution equations for higher order PDFs has to be considered.

If one compares the Oboukhov model (2) with the evolution equation for the single-particle PDF (3), one immediately recognizes that this model is based on a simple closure assumption. In the following, we shall present an exact generalization of this equation starting from (3) and (6). To this end, we introduce the conditional probability distribution $p(\mathbf{v}', \mathbf{r}', t'; \mathbf{v}, \mathbf{r}, t|\mathbf{u}, \mathbf{x}, \mathbf{y}; t')$, $f(\mathbf{v}', \mathbf{r}', t'; \mathbf{v}, \mathbf{r}, t; \mathbf{u}, \mathbf{x}, \mathbf{y}, t') = p(\mathbf{v}', \mathbf{r}', t'; \mathbf{v}, \mathbf{r}, t|\mathbf{u}, \mathbf{x}, \mathbf{y}; t')\times f(\mathbf{u}, \mathbf{x}, \mathbf{y}, t')$ and solve Eq. (6) for $f(\mathbf{v}, \mathbf{r}, t; \mathbf{u}, \mathbf{x}, \mathbf{y}, t)$:

$$f(\mathbf{v}, \mathbf{r}, t; \mathbf{u}, \mathbf{x}, \mathbf{y}, t) = f_0(\mathbf{v}, \mathbf{r}, t; \mathbf{u}, \mathbf{x}-\mathbf{u}t, \mathbf{y}, 0) - \int_0^t dt' [\nabla_{\mathbf{u}} \int d\mathbf{v}'\, d\mathbf{r}'\, \mathbf{A}(\tilde{\mathbf{x}}-\mathbf{r}', \mathbf{v}') p(\mathbf{v}', \mathbf{r}', t'; \mathbf{v}, \mathbf{r}, t|\mathbf{u}, \tilde{\mathbf{x}}, \mathbf{y}, t') f(\mathbf{u}, \tilde{\mathbf{x}}, \mathbf{y}, t')]_{\tilde{\mathbf{x}}=\mathbf{x}-\mathbf{u}(t-t')}. \tag{7}$$

The first term stems from the initial condition. Inserting this expression into Eq. (3), one obtains the following evolution equation for the single-particle PDF:

$$\dot{f}(\mathbf{u}, \mathbf{x}, \mathbf{y}, t) + \mathbf{u}\cdot\nabla_{\mathbf{x}} f(\mathbf{u}, \mathbf{x}, \mathbf{y}, t) = -\nabla_{\mathbf{u}}\cdot\int_0^t dt'\, D^1(\mathbf{u}, t, t'|\mathbf{u}, \mathbf{x}-\mathbf{u}(t-t'), \mathbf{y}) f(\mathbf{u}, \mathbf{x}-\mathbf{u}(t-t'), \mathbf{y}, t') + \nabla_{\mathbf{u}} \int_0^t dt'\, D^2(\mathbf{u}, t, t'|\mathbf{u}, \mathbf{x}-\mathbf{u}(t-t'), \mathbf{y}) [\nabla_{\mathbf{u}} f(\mathbf{u}, \tilde{\mathbf{x}}, \mathbf{y}, t')]_{\tilde{\mathbf{x}}=\mathbf{x}-\mathbf{u}(t-t')} + f_{\mathrm{init}}. \tag{8}$$

We have arrived at a generalized Fokker-Planck equation, i.e., an equation including a time memory. However, the drift and diffusion terms depend on the (unknown) conditional PDF $p(\mathbf{v}', \mathbf{r}', t'; \mathbf{v}, \mathbf{r}, t|\mathbf{u}, \mathbf{x}, \mathbf{y}, t')$. Furthermore, a term f_{init} stemming from initial conditions is introduced by solving Eq. (6). Such a term typically arises for non-Markovian stochastic processes.

Let us now estimate the generalized drift and diffusion terms. The generalized diffusion term is explicitly given by

$$D^2(\mathbf{u}, t, t'|\mathbf{u}, \mathbf{x}, \mathbf{y}) = \int d\mathbf{v}'\, d\mathbf{r}'\, d\mathbf{v}''\, d\mathbf{r}'' [\mathbf{A}(\mathbf{x}-\mathbf{r}', \mathbf{v}')\mathbf{A}(\mathbf{x}-\mathbf{r}-\mathbf{r}'', \mathbf{v}'') p(\mathbf{v}', \mathbf{r}', t; \mathbf{v}'', \mathbf{r}'', t'|\mathbf{u}, \mathbf{x}, \mathbf{y}; t')]_{\mathbf{r}=\mathbf{u}(t-t')}. \tag{9}$$

This term is the conditional two-time, two-scale acceleration-acceleration correlation:

$$D^2(\mathbf{u}, t, t'|\mathbf{u}, \mathbf{x}, \mathbf{y}) = \langle a(\mathbf{x}, t) a(\mathbf{x}-\mathbf{r}, t')|\mathbf{U}(t', \mathbf{y}), \mathbf{X}(t', \mathbf{y})\rangle_{\mathbf{r}=\mathbf{u}(t-t')}. \tag{10}$$

Here, $\mathbf{U}(t, \mathbf{y})$ and $\mathbf{X}(t, \mathbf{y})$ is the Lagrangian velocity and position of the fluid particle under consideration. The drift term is obtained according to

$$D^1(\mathbf{u}, t, t'|\mathbf{u}, \mathbf{x}, \mathbf{y}) = [\nabla_u\cdot\langle a(\mathbf{x}, t) a(\mathbf{x}-\mathbf{r}, t')|\mathbf{U}(t', \mathbf{y}), \mathbf{X}(t', \mathbf{y})\rangle]_{\mathbf{r}=\mathbf{u}(t-t')}. \tag{11}$$

Thus far, all the calculations are exact. We shall now introduce an approximation which allows one to relate the diffusion term to Eulerian characteristics of the turbulent acceleration field at the marker position $\mathbf{X}(t, \mathbf{y})$. To this end, we neglect the dependency of the conditional expectation values (10) and (11) on the behavior of the marker particle.

Formally, this is achieved by the approximation

$$p(\mathbf{v}', \mathbf{r}', t'; \mathbf{v}, \mathbf{r}, t|\mathbf{u}, \mathbf{x}, \mathbf{y}; t) \approx f_E(\mathbf{v}', \mathbf{r}', t'; \mathbf{v}, \mathbf{r}, t). \quad (12)$$

The drift term (11) vanishes, whereas the diffusion term (10) equals the two-point two-time acceleration-acceleration correlation $\langle \mathbf{a}(\mathbf{x}, t)\mathbf{a}(\mathbf{x} - \mathbf{r}, t')\rangle_{\mathbf{r}=\mathbf{u}(t-t')}$. The approximation (12) means that the correlation between acceleration fields at space time points $(t, \mathbf{x})$ and $(t', \mathbf{x} - \mathbf{r})$ is independent on the velocity $\mathbf{u}$ of a marker particle at point $(t', \mathbf{x} - \mathbf{r})$. This approximation (12) leads us to the Fokker-Planck equation including a memory term:

$$\dot{f}(\mathbf{u},\mathbf{x},\mathbf{y};t) + \mathbf{u}\cdot\nabla_{\mathbf{x}} f(\mathbf{u},\mathbf{x},\mathbf{y};t) = \nabla_{\mathbf{u}} \int_0^t dt' D^2(\mathbf{u},t,t')[\nabla_{\mathbf{u}} f(\mathbf{u},\tilde{\mathbf{x}},\mathbf{y};t')]_{\tilde{\mathbf{x}}=\mathbf{x}-\mathbf{u}(t-t')}. \quad (13)$$

This is a generalization of Oboukhov's model since long range correlations in the Eulerian acceleration field are taken into account.

The equal time acceleration-acceleration correlation has been studied in various publications [11–14]. However, no information seems to be available for $t \neq t'$. Therefore, we are forced to resort to dimensional analysis on the basis of Kolmogorov's theory K41 [2]. Since we are interested in the behavior in the inertial range, only the gradient pressure contribution to the acceleration is taken into account. According to the standard theory of Kolmogorov [2], the acceleration-acceleration correlation function $\langle \mathbf{a}(\mathbf{x}, t)\mathbf{a}(\mathbf{x} + \mathbf{r}, t')\rangle$ has the form $(\epsilon_0^{3/2}/\nu^{1/2}) \times F\{[\mathbf{r}/(\nu^{3/4}/\epsilon^{1/4})], [(t - t')/(\nu^{1/2}/\epsilon^{1/2})]\}$, where ϵ_0 denotes the local mean energy dissipation rate. In the inertial range of turbulence, this expression should be independent on the kinematic viscosity ν which leads to the functional dependency $\langle \mathbf{a}(\mathbf{x}, t)\mathbf{a}(\mathbf{x} + \mathbf{r}, t')\rangle = (\epsilon_0^{4/3}/r^{2/3})H[\mathbf{r}/(t - t')^{3/2}]$, where H is an arbitrary function with $H(\infty) = \text{const}$. As a result, the diffusion term takes the form

$$D^2_{ij}(\mathbf{u}, \tau) = \frac{\epsilon_0^{3/2}}{(\tau u)^{2/3}}\left[\delta_{ij}\alpha\left(\frac{u}{\tau^{1/2}}\right) + \frac{u_i}{u}\frac{u_j}{u}\beta\left(\frac{u}{\tau^{1/2}}\right)\right]. \quad (14)$$

Furthermore, one can formally show that the generalized Fokker-Planck equation possesses solutions with scaling behavior $1/t^6 f(\mathbf{u}/t^{1/2}, \mathbf{x}/t^{3/2}, \mathbf{y}/t^{3/2})$ similar to the PDFs belonging to Oboukhov's model. Usually, such a scaling behavior is thought to be intimately related to Gaussian statistics. In the following, we shall indicate that this need not be the case.

Let us consider the velocity PDF $h(\mathbf{u}, t)$, which due to isotropy is a function of $u = |\mathbf{u}|$. It obeys the equation (in d dimensions)

$$\dot{h}(\mathbf{u}; t) = \frac{1}{u^{d-1}}\frac{\partial}{\partial u}\int_0^t \frac{dt'}{(t - t')^{2/3}} u^{d-1-2/3} Q_0\left(\frac{u}{\sqrt{t - t'}}\right) \times \frac{\partial}{\partial u} h(\mathbf{u}; t'). \quad (15)$$

Since we are not able to give a more detailed form of the function $Q_0(\xi) = \epsilon_0^{3/2}[\alpha(\xi) + \beta(\xi)]$, we assume that it has a power law behavior in ξ, $Q_0(\xi) = Q_0\xi^\nu$, $\nu > 0$. The integral in (15) diverges at $t' = t$ for $2/3 + \nu/2 > 1$. A regularized version is

$$\dot{h}(\mathbf{u}; t) = \frac{1}{u^{d-1}}\frac{\partial}{\partial u} u^{d-1-2/3+\nu} Q_0 \frac{\partial}{\partial u} D_t^p h(\mathbf{u}; t)$$
$$p = \nu/2 - 1/3, \quad (16)$$

where we use the Riemann-Liouville derivative, $D_t^p f(t) = \{1/[\Gamma(n-p)]\}(d^n/dt^n)\int_0^t\{(dt)/[(t-t')^{1-n+p}]\}f(t')$, $(n-1<p<n)$ [15]. The case $\nu = 2/3$ is just the Oboukhov model with a Gaussian distribution. For the following, let us assume that $n=1$, i.e., $1/3<\nu/2<4/3$. For this class the behavior of the velocity is described by the so-called continuous time random walk model, where, in the present case, the diffusion term is velocity dependent [15]. An analytical solution to this problem is known [16] and the PDF is positive (which is not guaranteed, e.g., for $n=0$). The PDF takes the form

$$h(v, t) = \int_0^\infty ds\, n(s, t) h_1(u, s, 2/3 - \nu). \quad (17)$$

Here, $n(s,t) = \frac{1}{\alpha}(t/s^{1+1/\alpha})l_\alpha(t/s^{1/\alpha})$ and $l_\alpha(x)$ is a one-sided Lévy distribution whose Laplace transform is $l_\alpha(y) = \exp -y^\alpha$ $(\alpha = \nu/2 - 1/3)$. The probability distribution $h_1(u, s, \Theta)$ is the PDF of Eq. (16) with $p = 0$ [17]:

$$h_1(u, s, \Theta) = \frac{A(\Theta, d)}{[Q_0(2 + \Theta)^2 s]^{d/(2+\Theta)}}\exp\left[-\frac{u^{(2+\Theta)}}{Q_0(2 + \Theta)^2 s}\right]. \quad (18)$$

The shape of the obtained probability distribution is a function of the parameter ν. However, ν may be fixed by fitting the theoretical curve to the experimentally determined PDF. Figure 1 exhibits the result of the fits to the data of the Lyon group [6]. Each PDF can be well reproduced by fixing a definite value of ν. This supports the power law ansatz for $Q(\xi)$. We mention that the introduction of a t dependence of the value ν allows us to parametrize intermittency effects related with a change of the shape of the PDF. However, this is not the topic of the present Letter. Our aim was to investigate a closure assumption on the hierarchy of evolution equations for N point PDFs leading to velocity PDFs with fat tails exhibiting simple scaling behavior $h(u, t) = 1/t^{3/2}h(u/\sqrt{t})$.

In summary, we have investigated the evolution of the PDF of a marker particle in a fully developed homogeneous and isotropic turbulent fluid flow. As an exact result, we have obtained a generalized Fokker-Planck equation including a temporal memory. The simplest possible closure approximation has led us to a generalized Fokker-Planck equation with vanishing drift term, whereas the diffusion term is related to the Eulerian two-point two-time acceleration-acceleration correlation. A K41-scaling assumption on this quantity yields the well-known scaling behavior for the Lagrangian PDF. An assumed power law behavior of the diffusion coefficient $Q_0(\xi) \approx \xi^\nu$ has led us to a velocity statistics belonging to the class of continuous time random walks [15]. These are non-Markovian generalizations of the simple random walk underlying Oboukhov's model. A suitable choice of the free parameter ν allows us to reproduce the experimentally determined PDFs, which for small time differences are strongly non-Gaussian with fat tails. Finally, we want to mention that our results support the ideas of [7], who also emphasize the importance of memory effects and establish a non-Markovian model for the behavior of the acceleration. However, the process devised in [7] is a multifractal random walk, not a continuous time random walk with velocity dependent diffusion coefficient, as obtained by our analysis.

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