Classifying *N*-Qubit Entanglement via Bell's Inequalities

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All the states of *N* qubits can be classified into $N - 1$ entanglement classes from 2-entangled to *N*-entangled (fully entangled) states. Each class of entangled states is characterized by an entanglement index that depends on the partition of *N*. The larger the entanglement index of a state, the more entangled or the less separable is the state in the sense that a larger maximal violation of Bell's inequality is attainable for this class of state.

Bell's inequalities [1] were initially dealing with two qubits, i.e., two-level systems. They ruled out various kinds of local hidden variable theories. Recently with the emergence of the field of quantum information where the entangled states are essential, Bell's inequalities provide also a necessary criterion for the separability of 2-qubit states. This is because Bell's inequalities are observed by all separable 2-qubit states. For pure states Bell's inequalities are also sufficient for separability [2]. The more entangled the state, the larger is the maximal violation of Bell's inequalities.

On the one hand, Bell's inequalities were generalized to *N* qubits [3–5], whose violations provide a criterion to distinguish the totally separable states from the entangled states. On the other hand, with the experimental realization of multiparticle entanglement [6], Bell's inequalities in terms of the Mermin-Klyshko (MK) polynomials [3,7] were generalized to the case of $N = 3$ for the detection of fully entangled 3-qubit states [8] and to the *N*-qubit case for the detection of fully entangled *N*-qubit states [9,10]. And these results about full entanglement are inferred by the quadratic Bell inequalities [11].

In this Letter we provide a detailed classification of various types of *N*-qubit entanglement from total separability to full entanglement based on the amount of violation of Bell inequalities. It turns out that an entanglement index can be defined to characterize the entanglement class. Before presenting our classification of *N*-qubit entanglement we first review the classification of 2-qubit and 3-qubit entanglement by Bell's inequalities.

Quadratic Bell's inequalities for 2-qubit system.—First let us consider a system of two qubits labeled by *A* and *B*. There are only two types of states, separable or entangled. If a 2-qubit state ρ is separable, i.e., a pure product state or a mixture of pure product states, the well-known Bell-CHSH (Clauser, Horne, Shimony, and Holt) inequality [12]

$$\langle AB + AB' + A'B - A'B' \rangle_\rho \leq 2 \qquad (1)$$

holds true for all testing observables $A^{(\prime)} = \vec{a}^{(\prime)} \cdot \vec{\sigma}_A$ and

$B^{(\prime)} = \vec{b}^{(\prime)} \cdot \vec{\sigma}_B$. Here σ_A and σ_B are the Pauli matrices for qubits *A* and *B*, respectively; the norms of the real vectors $\vec{a}^{(\prime)}, \vec{b}^{(\prime)}$ are equal to 1; $\langle AB \rangle_\rho = \text{Tr}(\rho AB)$ denotes the average of the observable *AB* in the state ρ as usual.

If the state ρ is entangled, then the upper bound of the average of the Bell operator is $2\sqrt{2}$ [13], which is attainable for maximally entangled states. For the present purpose this upper bound is expressed more properly in a quadratic form of Bell's inequality [11]

$$\langle X \rangle_\rho^2 + \langle Y \rangle_\rho^2 \leq 4 \qquad (2)$$

in terms of two observables $X = A'B + AB'$ and $Y = AB - A'B'$ instead of one Bell operator $X + Y$. This inequality is satisfied by all 2-qubit states, entangled or separable. For separable states the Bell-CHSH inequality (1) can also be equivalently expressed as

$$|\langle X \rangle_\rho| + |\langle Y \rangle_\rho| \leq 2 \qquad (3)$$

in terms of observables *X* and *Y* because of the arbitrariness of the testing observables $A^{(\prime)}$ and $B^{(\prime)}$.

To summarize, there is only one entanglement class of 2-qubit states, i.e., 2-entangled states. For the separable states the Bell-CHSH inequality is observed while for the entangled states the Bell-CHSH inequality can be violated. By regarding $\langle X \rangle_\rho$ and $\langle Y \rangle_\rho$ as two axes of a plane, respectively, we can put all the results known so far in a diagram as shown in Fig. 1.

Classification of 3-qubit entanglement.—Now we consider three qubits labeled by *A*, *B*, and *C*. There are three types of 3-qubit states: (i) totally separable states denoted as $(1_3) = $ {mixtures of states of form$\rho_A \otimes \rho_B \otimes \rho_C$}, (ii) 2-entangled states denoted as $(2, 1) = $ {mixtures of states of form$\rho_A \otimes \rho_{BC}, \rho_{AC} \otimes \rho_B, \rho_C \otimes \rho_{AB}$}, and (iii) fully entangled states denoted as $(3) = \{\rho_{ABC}\}$ including the Greenberger-Horn-Zeilinger (GHZ) state [14].

These three types of 3-qubit states can be discriminated from one another by the averages of the MK polynomials defined as
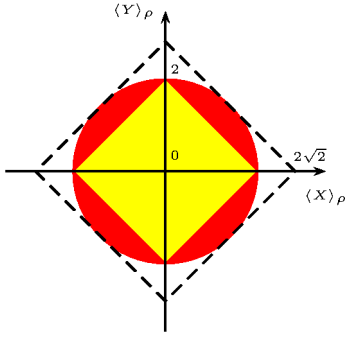
FIG. 1 (color online). All separable states lie in the inner square $|\langle X \rangle_\rho| + |\langle Y \rangle_\rho| \leq 2$ while a general 2-qubit state, whether separable or not, is bounded by a circle $\langle X \rangle_\rho^2 + \langle Y \rangle_\rho^2 \leq 4$ instead of the dashed square $|\langle X \rangle_\rho| + |\langle Y \rangle_\rho| \leq 2\sqrt{2}$.

$$F_3 = (AB' + A'B)C + (AB - A'B')C', \qquad (4a)$$

$$F_3' = (AB' + A'B)C' - (AB - A'B')C, \qquad (4b)$$

where C and C' are two observables of the third qubit defined similarly to A and B. In fact, for totally separable states the Bell-Klyshko inequality reads

$$\max\{|\langle F_3 \rangle_\rho|, |\langle F_3' \rangle_\rho|\} \leq 2 \quad \text{if } \rho \in (1_3), \qquad (5)$$

whose violation ensures a nonseparable state which can be either a 2-entangled state or a fully entangled state. The maximal violation is different for 2-entangled states and fully entangled states [11]

$$\langle F_3 \rangle_\rho^2 + \langle F_3' \rangle_\rho^2 \leq 2^3 \quad \text{if } \rho \in (2, 1), \qquad (6a)$$

$$\langle F_3 \rangle_\rho^2 + \langle F_3' \rangle_\rho^2 \leq 2^4 \quad \text{if } \rho \in (3). \qquad (6b)$$

Therefore the violation of inequality Eq. (6a) ensures a fully entangled state.

In summary, there are two different entanglement classes of 3-qubit states, namely, 2-entangled states and fully entangled states. Every class of entangled states can give rise to a different violation of the Bell-Klyshko inequality. We can put all these known results into a diagram as Fig. 2 by regarding the averages of observables F_3 and F_3' as two axes of a plane.

Classification of N-qubit entanglement.—Now we turn to the classification of N-qubit states. The number of the types of N-qubit states is the same with the number of all partitions $(\vec{n}) = (n_1, n_2, \ldots, n_N)$ of N with n_i ($i = 1, \ldots, N$) being integers such that

$$\sum_{i=1}^{N} n_i = N, \qquad (N \geq n_1 \geq n_2 \geq \ldots \geq n_N \geq 0). \quad (7)$$

In fact, the partition $(\vec{n})$ is in a one-to-one correspondence with the types of states that are mixtures of states $\rho_{n_1} \otimes \rho_{n_2} \otimes \cdots \otimes \rho_{n_N}$, where ρ_{n_i} is a fully entangled state of any n_i qubits ($i = 1, \ldots, N$) [15]. Therefore we can label different types of N-qubit states by different parti-
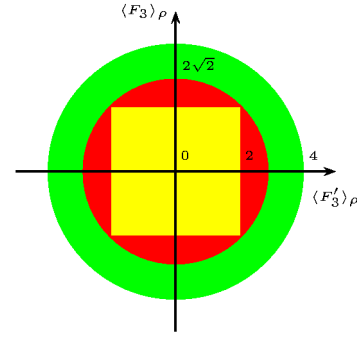

FIG. 2 (color online). Classification of 3-qubit entanglement. Separable states reside inside the square while the 2-entangled states are bounded by the smaller circle. If the averages of F_3 and F_3' in one state are outside the smaller circle, then the state is fully entangled.

tions of N. For example, the partition (N) corresponds to the fully entangled states while the partition $(1, 1, \ldots, 1) = (1_N)$ corresponds to the totally separable states. As a result the number of different types of N-qubit states is also the number of all the irreducible representations of the permutation group with N elements. This is not a coincidence because the types of the states as well as the MK polynomials defined below are invariant under the permutations of qubits.

To classify all types of N-qubit states, we employ the MK polynomials for the system of the N qubit. If we define $F_2 = X + Y$ and $F_2' = X - Y$ the MK polynomials are defined recursively as

$$F_N = \frac{1}{2}(D_N + D_N')F_{N-1} + \frac{1}{2}(D_N - D_N')F_{N-1}' \qquad (8)$$

for $N \geq 3$, where D_N and D_N' are observables of the Nth qubit and F_N and F_N' are MK polynomials for the $(N - 1)$ qubit. And observable F_N' is defined similarly to F_N with the primed and unprimed observables interchanged. The averages of these two observables in totally separable states satisfy the Bell-Klyshko inequality

$$\max\{|\langle F_N \rangle_\rho|, |\langle F_N' \rangle_\rho|\} \leq 2 \quad \text{if } \rho \in (1_N). \qquad (9)$$

As we will see immediately some types of states are more entangled than other types in the sense that a larger violation of this inequality is attainable. And the upper bound of the violation of the state in $(\vec{n})$ is related to *the entanglement index* defined as

$$E_N(\vec{n}) = N - K_1(\vec{n}) - 2L(\vec{n}) + 2, \qquad (10)$$

where $L(\vec{n})$ is the number of entries in $(\vec{n})$ that are greater than or equal to 2 and $K_1(\vec{n})$ is the number of entries in $(\vec{n})$ that are equal to 1, i.e., the number of separated single qubits. Here the totally separable states are excluded, i.e., $K_1 < N$. In other words, $N - K_1(\vec{n})$ is exactly the number of entangled qubits while $L(\vec{n})$ is exactly the number of groups into which the entangled $N - K_1$ qubits are divided with each group of qubits fully entangled.

Obviously the entanglement index is an integer satisfying $2 \leq E_N(\vec{n}) \leq N$.

For example, in the case of $N = 4$ we have five partitions, therefore five types of states: (i) fully entangled 4-qubit states (4) with $L(4) = 1$, $K_1(4) = 0$, and $E_4(4) = 4$; (ii) states of a group of a fully entangled 3-qubit and a separated single qubit (3, 1) with $L(3, 1) = 1$, $K_1(3, 1) = 1$, and $E_4(3, 1) = 3$; (iii) (2, 2) stands for the states of two groups of an entangled 2-qubit with $L(2, 2) = 2$, $K_1(2, 2) = 0$, and $E_4(2, 2) = 2$; (iv) $(2, 1, 1) = (2, 1_2)$ corresponds to the state of an entangled 2-qubit together with two separated qubits with $L(2, 1_2) = 1$, $K_1(2, 1_2) = 2$, and $E_4(2, 1_2) = 2$; (v) $(1, 1, 1, 1) = (1_4)$ corresponds to the totally separable states with $L(1_4) = 0$, $K_1(1_4) = 4$, and $E_4(1_4) = 2$.

According to the entanglement index all the N-qubit states can be classified into N entanglement classes: the class of totally separable states S_1 and the class of E-entangled states $\rho_E \in S_E$ ($E = 2, \ldots, N$), which are mixtures of entangled states with the same entanglement index E:

$$\rho_E = \sum_{E_N(\vec{n})=E} p(\vec{n}) \rho_{n_1} \otimes \rho_{n_2} \otimes \cdots \otimes \rho_{n_N}, \quad (11)$$

where the summation is over all states corresponding to the same partition and all partitions with the same entanglement index E and $p(\vec{n})$ is a probability distribution. One can also define a separability index as $S_N(\vec{n}) = 2L(\vec{n}) + K_1(\vec{n})$ with $S_N + E_N = N + 2$. An E-entangled state is also an S-separable state. The separability index S_N can be regarded as the effective number of qubits that can realize the partition $(\vec{n})$ since there are at least two qubits in each group of fully entangled qubits. Now we are ready to present our main result.

Classification Theorem: *The N-qubit states are classified into $N - 1$ entanglement classes and the E-entangled states satisfy the following quadratic Bell inequality:*

$$\langle F_N \rangle_\rho^2 + \langle F_N' \rangle_\rho^2 \leq 2^{E+1} \quad \text{if } \rho \in S_E. \quad (12)$$

Therefore the larger the entanglement index, the larger is the maximal violation of the Bell-Klyshko inequality. In this sense the larger the entanglement index, the more entangled or the less separable is the state.

We postpone the proof to the next section. The upper bound of the inequality for the E-entangled states is attainable for the pure state of type $(n_1, n_2, \ldots, n_N)$ that is a product of maximal entangled states for n_i qubits where for $n_i \geq 3$ the maximal entangled state is chosen as the GHZ state and for $n_i = 2$ the maximal entangled state is chosen as one of the Bell states. The two testing observables for each qubit can be simply chosen as σ_x and σ_y.

The largest entanglement index $E_N(N) = N$ is reached by the partition (N). Therefore the N-entangled states or fully entangled states can be distinguished from other classes of states by Bell's inequality. The $(N - 1)$-entangled states are states of one group of entangled $N - 1$ qubits and a separated single qubit. The $(N - k)$-entangled states are states of a group of fully entangled $(N - k)$ qubits and a group of fully entangled k qubits with $k \geq 2$. Thus the states with different k have the same property of nonseparability. The least entanglement index is possessed by the 2-entangled states.

We notice that when $K_1 = 0$ there is no separated qubits and the more entries in the partition $(\vec{n})$, i.e., the more groups into which the entangled qubits are divided, the more separable is the state. Conversely for states with larger entanglement index, the N qubits are divided into less groups of fully entangled qubits.

The classification of N-qubit entanglement can also be put into a diagram as in Fig. 3 by regarding the average of F_N and F_N' as two axes of a plane. Here $N - 1$ circles and one square are the boundary for all kinds of N-qubit entanglement.

Proof of the theorem.—At first we notice that the theorem is true when $N = 3$ where we have $E_3(3) = 3$ and $E_3(2, 1) = 2$. Since an E-entangled state ρ_E of N qubits is a convex mixture of states in $(\vec{n})$ with the entanglement index $E_N(\vec{n}) = E$, it is enough to prove the inequality in the theorem for a general state ρ corresponding to the partition $(\vec{n})$. For convenience we denote $\mathcal{B}_N(\vec{n}) = \langle F_N \rangle_\rho^2 + \langle F_N' \rangle_\rho^2$. Further if the smallest entry of partition $(\vec{n})$ is $k \geq 1$ then we denote $(\vec{n}) = (\vec{n}_k, k)$, where $(\vec{n}_k)$ is a partition of $N - k$ whose smallest elements are greater or equal to k.

If the state ρ is of type $(\vec{n}_1, 1)$, then there is at least one separated single qubit. Because the MK polynomials are symmetric under the permutation of qubits, it is enough to suppose that the Nth qubit is separated. From the definition Eq. (8) of the MK polynomials we obtain
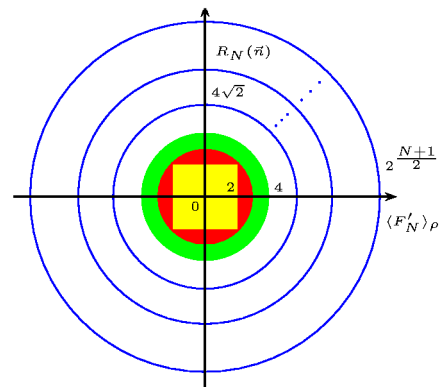

FIG. 3 (color online). Classification of N-qubit entanglement. While all separable states are bounded in the square, the states of type $(\vec{n})$ are bounded within a circle of a radius $R_N = 2^{[E_N(\vec{n})+1]/2}$. The largest circle with a radius of $2^{(N+1)/2}$ corresponds to the fully entangled states.

$$\mathcal{B}_N(\vec{n}_1, 1) = \tfrac{1}{2}(\langle D_N \rangle_\rho^2 + \langle D_N' \rangle_\rho^2)\mathcal{B}_{N-1}(\vec{n}_1) \le \mathcal{B}_{N-1}(\vec{n}_1). \tag{13}$$

Obviously this inequality is attainable. If the state ρ is of type $(\vec{n}_k, k)$ with $k \ge 2$, then there is at least one group of a fully entangled k-qubit. Without loss of generality we suppose the last k qubits are separated from other $N - k$ qubits. By rewriting the MK polynomial as [4]

$$F_N = \tfrac{1}{4}[F_{N-k}(F_k + F_k') + F_{N-k}'(F_k - F_k')], \tag{14}$$

where F_k and F_k' are MK polynomials for the last k qubits and F_{N-k} and F_{N-k}' are MK polynomials for the remaining $N - k$ qubits, we obtain

$$\mathcal{B}_N(\vec{n}_k, k) = \tfrac{1}{8}\mathcal{B}_k(k)\mathcal{B}_{N-k}(\vec{n}_k) \le 2^{k-2}\mathcal{B}_{N-k}(\vec{n}_k), \tag{15}$$

where we have used the quadratic Bell inequality for the fully entangled k-qubit state $\mathcal{B}_k(k) \le 2^{k+1}$ [11], which is attainable for the GHZ state for k qubits. Therefore the above inequality is also attainable.

By denoting K_k as the number of k's as entries of the partition $(\vec{n})$, we can rewrite the partition as $(\vec{n}) = (M_{K_M}, \ldots, 2_{K_2}, 1_{K_1})$, where M is the largest entry of $(\vec{n})$. As a result

$$\mathcal{B}_N(\vec{n}) = \mathcal{B}_N(M, M_{K_M-1}, (M-1)_{K_{M-1}}, \ldots, 3_{K_3}, 2_{K_2}, 1_{K_1}) \le 2^{K_3(3-2)}2^{K_4(4-2)}\cdots 2^{K_{M-1}[(M-1)-2]}2^{(K_M-1)(M-2)}\mathcal{B}_M(M)$$
$$\le 2^{3+\sum_{k=3}^{M} K_k(k-2)} = 2^{N+3-2L-K_1} = 2^{E_N(\vec{n})+1}, \tag{16}$$

where identities $N = \sum_{k=1}^{M} kK_k$ and $L = \sum_{k=2}^{M} K_k$ have been used. The first inequality is a consequence of applying inequality Eq. (13) K_1 times and Eqs. (15) K_k times for $k = 3, 4, \ldots, M - 1$ and $K_M - 1$ times when $k = M$. The second inequality stems from the quadratic Bell's inequality for fully entangled M-qubit states. Since all the inequalities in the proof are attainable, the upper bound for the E-entangled states is also attainable.

Conclusions and discussions.—In this Letter we have provided a classification of N-qubit entanglement by introducing the integer entanglement index E_N. Some orders have been established among various types of N-qubit entanglement: the larger the entanglement index, the larger maximal violation of the Bell's inequality is attainable and therefore more entangled or less separable is the state.

However, our classification is far from being complete regarding the following two connected aspects. First, as Bell's inequality provides only a sufficient criterion for the entanglement, the hierarchy of quadratic Bell inequalities for different classes of entangled states is sufficient for the detection of E entanglement. Second, within one entanglement class there may exist states with different types of entanglement. These differences obviously cannot be detected by Bell's inequalities in terms of MK polynomials discussed here. By finding some observables other than the MK polynomials that are less symmetric, one may give a finer classification of N-qubit entanglement. It will be of interest to find some criteria to discriminate such degeneracy of the entanglement class.

Note added.—After the submission of the present work we became aware of the Letter by Nagata *et al.* [16], in which the same inequality (12) is proved. While this inequality was mainly employed to detect full entanglement in Ref. [16], we use the the same inequality to classify quantitatively all kinds of entanglement.

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