Topological Protection and Quantum Noiseless Subsystems

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Encoding and manipulation of quantum information by means of topological degrees of freedom provides a promising way to achieve natural fault tolerance that is built in at the physical level. We show that this topological approach to quantum information processing is a particular instance of the notion of computation in a noiseless quantum subsystem. The latter then provides the most general conceptual framework for stabilizing quantum information and for preserving quantum coherence in topological and geometric systems.

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Quantum information is sensitive: more bad things can happen to a quantum bit than to a classical bit [1]. A variety of schemes for protecting quantum information have been developed, including quantum error correction codes [2], decoherence-free subspaces [3], noiseless subsystems [4], bang-bang decoupling [5], and topological quantum computation [6]. The first four of these techniques are closely related to each other and can be described in a simple unified framework based on representations of the algebra of errors [4,7]. This paper shows that topological quantum computation also falls into the error-algebra framework. This result suggests that methods for preserving quantum coherence in general fall within a unified algebraic framework.

In the error-algebra framework quantum information is protected by using symmetry. The symmetry that protects quantum information can exist naturally in the interaction of the quantum information processing system with its environment, as in the case of decoherence-free subspaces and noiseless subsystems, the symmetry can be induced by adding additional dynamics as in the case of bang-bang decoupling, or the symmetry can exist implicitly as in quantum error correcting codes. The role of explicit, dynamical, and implicit symmetries in stabilizing quantum states and preserving quantum coherence, of course, goes far beyond quantum information processing: preservation of coherence via symmetry plays a role in virtually all quantum systems. One apparent exception to this rule is the case of topological quantum systems, in which topological degrees of freedom are intrinsically resilient to local errors. Although topological quantum computation is related to toric error correcting codes [8], the physical mechanism by which it preserves quantum information goes beyond toric codes. It is an interesting question, then, whether topological quantum computation, in particular, and topological quantum systems, in general, can be treated in a unified framework along with the above mechanisms. This Letter shows that the answer to this question is "yes."

First, review briefly the way in which symmetry protects quantum information [4,7]. Suppose that one has a quantum system S with Hilbert space $\mathcal{H} \cong \mathbb{C}^d$, interacting with an environment. The effect of the environment on the system is given by a set of error operators $\{E_\alpha\}$: each E_α represents some bad thing that can happen to the quantum system. Sums of arbitrary products of error operators together with their Hermitian conjugates generate the error algebra $\mathcal{A}$. This error algebra is the fundamental object in the algebraic approach to protecting quantum information: it contains all the information about the quantum information-stabilizing strategies. Let $\mathcal{A}' = \{X:[X, E_\alpha] = 0\}$ be the commutant of the error algebra. The elements of the unitary part $\mathcal{U}(\mathcal{A}')$ of $\mathcal{A}'$ are symmetries for the error algebra $\mathcal{A}$.

The degrees of freedom associated with observables in $\mathcal{A}'$ are *noiseless* ones [4,7]: they are by definition decoupled from the noise processes enacted by the elements of $\mathcal{A}$. These noiseless observables give rise to decoherence-free subspaces and noiseless subsystems as follows. It is a basic theorem of representations of algebras that the Hilbert space $\mathcal{H}$ then decomposes as follows:

$$\mathcal{H} \cong \oplus_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}, \qquad (1)$$

where the J label the different irreducible representations of the algebras $\mathcal{A}$ and $\mathcal{A}'$, d_J is the dimension of the Jth irreducible representation of $\mathcal{A}$, and n_J is the dimension of the Jth irreducible representation of $\mathcal{A}'$. [Formally, this decomposition of the Hilbert space into sums of tensor product spaces corresponds to the so-called central decomposition [9]: $\mathcal{A} \cong \oplus_{J\in\mathcal{J}} \mathbb{1}_{n_J} \otimes M(d_J, \mathbb{C})$, where M is the set of $d_J \times d_J$ matrices over $\mathbb{C}$.] The tensor product structure arises naturally because members of $\mathcal{A}$ and $\mathcal{A}'$ commute: in each term in the sum, the error operators in $\mathcal{A}$ act on the subsystem $\mathbb{C}^{d_J}$ while leaving the *noiseless subsystem* $\mathbb{C}^{n_J}$ unchanged. The decomposition (1) shows that nontrivial noiseless subsystems exist only when $\mathcal{A}$ has a noncommutative symmetry group $\mathcal{G} \equiv \mathcal{U}(\mathcal{A}')$. In the particular case in which $d_J = 1$, one has an instance of a noiseless code or decoherence-free subspace [3]. Bang-bang decoupling is a method for inducing an effective symmetry in the error dynamics that gives rise to effective noiseless subsystems. Finally, in [4], it was shown that this tensor product decomposition is at the root of quantum error correcting codes: errors act on the subsystem $\mathbb{C}^{d_J}$ while the quantum information lying in the encoded subsystem $\mathbb{C}^{n_J}$ remains unchanged. So virtually all known methods for protecting quantum information fall within the error-algebra formalism.

This formalism is also at the root of performing quantum information processing in a fault-tolerant fashion. Quantum manipulations within a noiseless subsystem can be performed by applying transformations from $\mathcal{A}'$. This last technique allows one to perform universal quantum computation using quantum logic gates (such as swap gates) that are not universal on the entire Hilbert space, a phenomenon known as encoded universality [10].

For what follows it is important to notice that the state-space structure (1) is reminiscent of *superselection* [11]. In superselection theory the algebra $\mathcal{A}$ is viewed as the one generated by the *whole* set of physical observables rather than the one associated with a set of distinguished interactions (the error operators). In this context $\mathcal{U}(\mathcal{A}')$ is called the gauge group. The operators of $\mathcal{A}$ are not able to change the quantum numbers associated with the gauge transformations and the state space accordingly splits in a direct sum of nonconnected sectors. Accordingly, the elements of the gauge group are operators that commute with *all* the physical quantities, and their eigenvalues therefore cannot be changed by any physical operation. It is well known that such a situation can occur only in the cases in which $\mathcal{A}$ describes an infinite set of degrees of freedom, the paradigmatic case being provided by field theory [12]. The different sectors describe now different inequivalent *phases* in which the system can exist; a major illustration of this state of affairs is provided by the phenomenon of spontaneous symmetry breaking.

Another very important occurrence of superselection is given by the quantization of systems whose classical configuration manifold $\mathcal{M}$ has nontrivial topology, e.g., with fundamental group $\pi_1(\mathcal{M}) \neq Id$ [13]. In this case the superselection sectors correspond to inequivalent quantizations labeled by irreducible representations of $\pi_1(\mathcal{M})$ [14]. When $\mathcal{M}$ is the manifold associated with N indistinguishable particles living in d dimensions, the different irreducible representations describe different quantum *statistics* [15]. For $d \geq 3$, $\pi_1(\mathcal{M})$ is given by the permutation group S_N, while for $d = 2$ the fundamental group is the braid group $\mathcal{B}_N$. Particles associated with one-dimensional irreducible representations of $\mathcal{B}_N$ are called Abelian *anyons,* and particles associated with higher dimensional representations are called non-Abelian anyons. This latter class of systems is exactly the one that has been argued to be useful for quantum computation [6]. Now we apply the error-algebra framework to topological information protection.

The prototype system in which we are interested is a lattice Λ having attached to each of its sites (or edges) a finite-dimensional quantum system, e.g., a qubit, with state space $\mathcal{H}_i$. The lattice Λ is supposed to be embeddable in a two-dimensional surface $\mathcal{M}$ with *genus g*. For example, the lattice could be a square lattice with periodic boundary conditions, embedded in a torus. The interaction among the local quantum systems is described by a *local* Hamiltonian H_Λ having a $D(g)$-dimensional degenerate ground state $C \subset \mathcal{H}_\Lambda \equiv \otimes_{i\in\Lambda}\mathcal{H}_i$, where D is an exponential function of its argument. C is the *code* subspace. The key idea is that H_Λ is designed in such a way that the ground-state degeneracy has a topological origin. This means that orthogonal elements of C have to correspond to different eigenvalues of global observables.

For some of the systems studied in [6,16] a complete set of commuting global observables in C can be constructed as follows. Let $\{\gamma\}_{i=1}^{2g}$ denote the set of noncontractible loops generating the homology group $H_1(\mathcal{M})$ of $\mathcal{M}$ [13]. One can consider the operators $X_\gamma \equiv \prod_{i\in\gamma} x_i$ where the $x_i \in \mathrm{End}(\mathcal{H}_i)$ are suitable site operators, e.g., σ_i^z. For the sake of simplicity assume that the X_γ's are Hermitian self-inverse operators, i.e., $X_\gamma^2 = \mathbb{1}$. If this is the case, C can be decomposed in terms of the 2^{2g} joint eigenvectors of the X_{γ_i} i.e., $C = \mathrm{span}\{|J\rangle \equiv |j_1, \ldots, j_{2g}\rangle\}$ where $X_{\gamma_i}|J\rangle = j_i|J\rangle\, (j_i \in \mathbf{Z}_2 \equiv \{-1, 1\})$.

We denote by $\mathcal{A}_{\mathrm{glob}}$ the Abelian algebra generated by the X_{γ_i}. Local operators $X \in \mathcal{A}_{\mathrm{loc}} \equiv \mathcal{A}'_{\mathrm{glob}}$ cannot by definition modify the global properties described by the X_{γ_i}'s. Local operators therefore (a) cannot induce tunneling between orthogonal ground states, and (b) cannot distinguish elements in the code subspace C. It follows that $\forall\ X \in \mathcal{A}_{\mathrm{loc}}$ one has $\langle J'|X|J\rangle = \delta_{J,J'}c(X)$, where $c: \mathcal{A}_{\mathrm{loc}} \mapsto \mathbb{C}$. If Π_C is the projector over the ground state C, a compact way to express the condition above is given by

$$\Pi_C X \Pi_C = c(X)\Pi_C, \qquad \forall\ X \in \mathcal{A}_{\mathrm{loc}}. \tag{2}$$

This latter relation amounts to saying that C behaves as an error correcting code with respect to the class of errors represented by local operators [2].

The vector space generated by the action of local operators over C comprises the whole of $\mathcal{H}_\Lambda$. Relations (2) above imply that the subspaces $\mathcal{A}_{\mathrm{loc}}|J\rangle$ for different J's are orthogonal and *isomorphic*. It follows that one has the following splitting according to the irreducible representations of $\mathcal{A}_{\mathrm{glob}}$:

$$\mathcal{H}_\Lambda = \oplus_{J\in\mathbf{Z}_2^{2g}} \mathcal{A}_{\mathrm{loc}}|J\rangle \cong C' \otimes C, \tag{3}$$

where C' is a $2^{\ln \dim\mathcal{H}_\Lambda - 2g}$-dimensional factor associated with *local* degrees of freedom. This factor is associated with the syndrome measurements for quantum error correction, and its local nature implies that those measurements can be performed with elements belonging to $\mathcal{A}_{\rm loc}$.

By comparing Eq. (3) with Eq. (1) it is not difficult to realize that the following holds.

Proposition.—The topologically protected sector C of $\mathcal{H}_\Lambda$ can be identified with a noiseless subsystem with respect to the error algebra generated by local interactions. The associated gauge group is generated by operators with nontrivial topological content.

It is quite important to make clear that all these properties are meant to hold in the limit in which the size $|\Lambda|$ goes to infinity; for finite size systems they are fulfilled only in an approximate way, though with exponential accuracy [6]. Formally Eq. (2) has then the meaning

$$\|\Pi_C X \Pi_C - c(X)\Pi_C\| = O(e^{-\alpha|\Lambda|^{1/n}}), \tag{4}$$

where $\alpha > 0$ and n is an integer. It is just when $|\Lambda| \mapsto \infty$ the $\mathcal{A}_{\rm loc}|J\rangle$ become truly disjointed sectors. In this limit one has an effective state-space splitting having a topological origin also known as homotopical superselection rule [14]. In the finite $|\Lambda|$ case there is always a (small) chance of a local perturbation inducing tunneling between different $|J\rangle$'s and of having different diagonal elements in C.

The result stated in the proposition above—in view of the general connections between error correcting codes and noiseless subsystems already established in [4,7]—is not conceptually totally surprising. On the other hand, it represents a novel and natural physics-based instance of those connections. Moreover, in this way we are also pointing out that, after tracing over the local degrees of freedom, i.e., C', one need not perform any active recovery from the error: a completely passive stabilization is achieved.

A very interesting situation is when our system is gapped; this means that there is a finite energy Δ between the ground and first excited states that remains finite even in the large size limit. In this case, every local modification of the ground state results in a finite increase of energy: the system is *incompressible*. Prototypes of this kind of systems are provided by fractional quantum Hall effect fluids and spin liquids [17].

In this case small perturbations $X \in \mathcal{A}_{\rm loc}$ are ineffective in inducing tunneling between the ground and the excited states (not just among ground states). Indeed, from elementary perturbation theory one has that the amplitude for those tunneling processes scale as $k/\Delta \ll 1$, where k is the typical strength of the matrix element of X between ground and excited states. Reasoning again in perturbation-theoretic fashion, it is clear that this relation along with Eq. (2) implies that ground-state degeneracy is *robust* against small and local perturbations. This is a signature of its topological nature. It has been recently argued that this kind of stability result might be generic for quantum spin systems in lattices with short range interactions [18]. Moreover, some (exotic) spin models whose ground states are robust for all weak enough local perturbations have been explicitly constructed [19].

Quantum information manipulation.—The primary purpose of this Letter is to identify in detail the connection between topological protection of quantum information and the error-algebra formalism of protecting quantum information via noiseless subsystems, quantum error correcting codes, etc. Now that that task has been performed, we use the general formalism developed above to address the important problem of the manipulation of topologically encoded quantum information. The actual way in which universal topological quantum computation is performed depends strongly on the underlying physical models [6,16]: the error-algebra formalism allows us to abstract certain common features of these models.

Within the described error-algebra framework, it is possible to describe how to perform quantum computation within the code subspace by creating local excitations and by moving them around the lattice. The key point here is that in the topological models we are examining the Hamiltonian spectrum comprising (non-Abelian) localized anyonic excitations [6]. By spatially exchanging those excitations one can enact operations that are able to induce coupling between different topological sectors. Moreover, these operations will depend just on some global, i.e., topological, feature of the exchange and therefore are stable against any local perturbation.

Acting on C with N-site operators, $X_{j_k}^{\alpha_k}$ creates an excited state with N local excitations (the α's label the different possible "colors"). These excited states are degenerate as long as the j's are kept far apart [6]. One can then build a degenerate subspace endowed with a bipartite (local and global) tensor product structure $\mathcal{H}_\alpha^N := \mathrm{span}\{\prod_{k=1}^N X_{j_k}^{\alpha_k} C \,/\, l \neq k \Rightarrow j_l \neq j_k\} \cong C'_\alpha \otimes C$. The basis states in $\mathcal{H}_J^N(\alpha)$ are labeled by the locations $j_1, \ldots, j_N$ of the N local excitations, e.g., anyon-antianyon pairs. In order to perform quantum manipulations, one resorts to the anyonic nature of these excited states. Excitations can be moved about the lattice, either by applying local dynamical swap operations [20] or by dragging them adiabatically, e.g., using an external potential [21], along some path with nontrivial braiding pattern b. Moving one excitation around another enacts an element of the braid group $\mathcal{B}_N$, which in turn performs a quantum logic operation on the quantum information registered in the code subspace. Finally, the excitations are annihilated (fusion): the result of the computation is registered in the local state of the system after fusion has taken place.

We denote by ρ the particular high-dimensional representation of the Braid group model involved in the given topological model. The sequence of excitation, braiding, and fusion can be schematically summarized by the following map chain:

$$C \stackrel{\prod_k X_k}{\longrightarrow} \mathcal{H}_\alpha^N \stackrel{\rho(b)}{\longrightarrow} \mathcal{H}_\alpha^N \stackrel{\prod_k X_k^\dagger}{\longrightarrow} C; \qquad (5)$$

it is important to note here that only the intermediate braiding step $\rho(\gamma)$ has a nontrivial topological content, and can therefore change the global quantum numbers.

The braid elements $\rho(\gamma)$ are expressible as an holonomy of a suitable statistical connection [15]; from this perspective the computational scheme sketched above provides a particular instance of the so-called holonomic approach to quantum computation [22]. In this kind of scheme, information is encoded in a degenerate eigenspace of a parametric family of (isodegenerate) Hamiltonians and manipulated by driving the parameters along suitable adiabatic paths. This enacts a transformation of the encoding space into itself via the holonomy associated with the Wilczek-Zee non-Abelian connection, i.e., gauge potential, generated by the Hamiltonian family [23]. In the topological case under examination the manifold of control parameters is given by the set of the coordinates of the anyonic excitations themselves. When the statistical connection has an holonomy group coinciding with a whole set of unitary transformations over C, the full computational power is achieved [22]. In this case universal fault-tolerant manipulations can be performed on the coding ground state C. (Such computation is an example of encoded universality.) The common holonomic nature of geometric and topological quantum computation suggests that conceptually there is a sort of continuous path from purely geometric to purely topological quantum information processing schemes. In order to optimize the fault-tolerance features one might think of designing non-Abelian Wilczek-Zee connections with maximal topological content.

Conclusions.—In this paper we discussed the relation between the topological approach to fault-tolerant quantum information processing and the quantum error correction-avoidance strategies. A unified view of this latter class of by-now standard techniques is provided by the algebraic notion of a noiseless subsystem. We showed that this notion is powerful enough to encompass even the former class: topologically protected quantum codes are an instance of noiseless subsystem. The crucial point consists in the separation of local and global degrees of freedom by means of the associated observable algebras. Moreover, we pointed out how information processing within this kind of noiseless subsystems is then achieved through the holonomic manipulations of (non-Abelian) anyonic excitations.

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