

Purification through Zeno-Like Measurements

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A series of frequent measurements on a quantum system (Zeno-like measurements) is shown to result in the “purification” of another quantum system in interaction with the former. Even though the measurements are performed on the former system, their effect drives the latter into a pure state, irrespectively of its initial (mixed) state, provided certain conditions are satisfied.

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It is well known that unstable particles or quantum states display a peculiar behavior at short and long times [1]. Phenomenologically, they are known to decay exponentially and this is well confirmed experimentally [2]. Short-time deviations were observed only very recently [3]. The deviations from the familiar exponential decay law are unavoidable consequences of the quantum dynamics both at short and long times, and the derivation of the exponential decay law itself is not a trivial matter in quantum mechanics. These deviations reflect the unitarity of the time-evolution operator or the time reversal symmetry of the Schrödinger equation at short times and the lower boundedness of the Hamiltonian or the stability of the vacuum at long times. See, e.g., Ref. [1] for a review.

The quantum behavior of unstable states at short times has been one of the central issues of investigation and discussion in recent years, since it is closely connected to the so-called quantum Zeno effect (QZE) [4,5], where the act of measurement [6] (usually represented by the von Neumann projection or the generalized spectral decomposition [7]) affects in an essential way the dynamics of the measured system and results in a hindrance of the decay process. The first attempt at the experimental observation of the QZE in an atomic transition process [8], following Cook’s theoretical work [9], has triggered heated discussions on this subject. Furthermore, another exciting experiment has been reported very recently: the observation of the QZE (and also of the inverse QZE [10]) in an atomic tunneling process [11], which is the first experimental observation of the (inverse) QZE in a truly unstable quantum system, unlike in the previous experiment [8] performed on an oscillating system.

In this Letter, we will shed new light on another (and so far not well explored) feature of the quantum dynamics with measurements, closely related to the QZE. Notice first that the system under consideration cannot be considered completely isolated and usually interacts with other systems. Therefore, it would be interesting and maybe more realistic to consider the case where the measurement, represented by a von Neumann projection for simplicity, is *not performed on the total system, but only on the system of interest*. Here, we consider such measurements and address the following point: *How does*

a series of frequent measurements on a system affect the dynamics of another system in interaction with the former? Under frequent measurements performed only on the former system, the latter evolves away from its initial state. We shall show that such measurements can result in a purification phenomenon. That is, a series of frequent measurements on system A, represented by projections on a given (usually pure) state of A, makes the state of system B, which interacts with A and is initially in *any* (mixed) state, approach a pure state, if certain conditions prescribed below are satisfied.

Let a total quantum system A + B be described by a Hamiltonian H of the form

$$H = H_A + H_B + H_{\text{int}}, \quad (1)$$

where $H_{A(B)}$ stands for a free Hamiltonian of system A(B) and H_{int} for an interaction. We prepare system A in its initial (pure) state $|\phi\rangle\langle\phi|$ at time $t = 0$. The initial state of system B, denoted by ρ_B , can be arbitrary. The initial state of the total system is

$$\rho_0 = |\phi\rangle\langle\phi| \otimes \rho_B \quad (2)$$

and its dynamics is governed by the Hamiltonian (1) unless it is interrupted by a series of measurements on system A, each of which is represented by a projection operator $\mathcal{O} = |\phi\rangle\langle\phi| \otimes \mathbb{1}$, performed at time intervals τ . Notice that this operator, even if it is a *bona fide* projection operator (we assume that $|\phi\rangle$ is normalized), does not return the time-evolved total system to its initial state. The projection is partial, in the sense that only the state of system A is set back to its initial state and that of system B is not initialized, even though the dynamics of B is certainly affected.

After N such measurements have been done, the survival probability of finding system A still in its initial state is represented by

$$\begin{aligned} P^{(\tau)}(N) &= \text{Tr}[(\mathcal{O}e^{-iH\tau}\mathcal{O})^N \rho_0 (\mathcal{O}e^{iH\tau}\mathcal{O})^N] \\ &= \text{Tr}_B[(V_\phi(\tau))^N \rho_B (V_\phi^\dagger(\tau))^N]. \end{aligned} \quad (3)$$

Notice that the quantity $V_\phi(\tau) \equiv \langle\phi|e^{-iH\tau}|\phi\rangle$ is an operator acting on the Hilbert space of system B. The density operators of the total and B systems read

$$\begin{aligned} \rho^{(\tau)}(N) &= (\mathcal{O}e^{-iH\tau}\mathcal{O})^N \rho_0 (\mathcal{O}e^{iH\tau}\mathcal{O})^N / P^{(\tau)}(N) \\ &= |\phi\rangle\langle\phi| \otimes \rho_B^{(\tau)}(N), \end{aligned} \quad (4a)$$

$$\rho_B^{(\tau)}(N) = (V_\phi(\tau))^N \rho_B (V_\phi^\dagger(\tau))^N / P^{(\tau)}(N), \quad (4b)$$

respectively. We collect only the right outcomes of measurements: this is implicit in the normalization factors in (4). Experimentally, this means that after each measurement, only those events will be retained in which system A has been found in its initial state.

In the ordinary situation, one performs infinitely frequent measurements by taking $N \rightarrow \infty$ and $\tau \rightarrow 0$, keeping $N\tau = T$, a finite nontrivial value; one easily checks that the ordinary QZE [1] appears in this case and the survival probability $P^{(\tau)}(N)$ increases as N becomes large, approaching unity in the $N \rightarrow \infty$ limit [5]. At the same time, the dynamics of system B becomes unitary in this limit, and this is an example of the so-called “quantum Zeno dynamics” [12]. However, we stress that our interest lies in a different situation: we keep the time interval τ between measurements finite and nonvanishing. If N were taken to be ∞ , the survival probability $P^{(\tau)}(N)$ would decay out completely for such $\tau \neq 0$, but we are interested in the asymptotic behavior of the state of system B for large but *finite* values of N . We expect that the effect of repeated measurements on system A would modify the dynamics of system B through its interaction with the measured system A, even if B has never been directly measured. To examine this idea, we need to clarify the asymptotic behavior of the state of system B, $\rho_B^{(\tau)}(N)$, for large N .

It is clear that the behavior of $\rho_B^{(\tau)}(N)$ is governed by the operator $V_\phi(\tau)$ in (4). Let us consider its eigenvalue problem. Since this operator is *not Hermitian*, $V_\phi^\dagger(\tau) \neq V_\phi(\tau)$, in general, we need to set up both the right- and left-eigenvalue problems

$$V_\phi(\tau)|u_n\rangle = \lambda_n|u_n\rangle, \quad \langle v_n|V_\phi(\tau) = \lambda_n\langle v_n|. \quad (5)$$

The eigenvalue λ_n is in general complex valued. Let us assume that the spectrum of the operator $V_\phi(\tau)$ is discrete and nondegenerate, and its eigenvectors form an orthonormal complete set in the following sense:

$$\sum_n |u_n\rangle\langle v_n| = \mathbb{1}, \quad \langle v_n|u_m\rangle = \delta_{nm}. \quad (6)$$

[It will soon become clear that the assumption of the nondegenerate spectrum is not essential for the following discussion except for that of the largest (in magnitude) eigenvalue λ_{\max} .] The operator itself is expanded in terms of its eigenvectors

$$V_\phi(\tau) = \sum_n \lambda_n |u_n\rangle\langle v_n|, \quad (7)$$

and we obtain

$$(V_\phi(\tau))^N = \sum_n \lambda_n^N |u_n\rangle\langle v_n|. \quad (8)$$

One can show [13] that the absolute value of the eigenvalue λ_n satisfies the inequality $0 \leq |\lambda_n| \leq 1$, $\forall n$, which reflects the unitarity of the time-evolution operator. It is now evident that, in the large N limit, the operator (8) is dominated by a single term

$$(V_\phi(\tau))^N \xrightarrow{\text{large } N} \lambda_{\max}^N |u_{\max}\rangle\langle v_{\max}|, \quad (9)$$

where $|u_{\max}\rangle$ and $\langle v_{\max}|$ are the eigenvectors belonging to λ_{\max} , provided the largest (in magnitude) eigenvalue λ_{\max} is unique, discrete, and nondegenerate.

Thus we reach the conclusion, under the assumption of unique, discrete, and nondegenerate λ_{\max} , that, in the large N limit with a nonvanishing τ , the state of system B in interaction with system A, on which N measurements are performed at time intervals τ , asymptotically approaches the pure state $|u_{\max}\rangle$

$$\rho_B^{(\tau)}(N) \xrightarrow{\text{large } N} |u_{\max}\rangle\langle u_{\max}| / (u_{\max}|u_{\max}), \quad (10)$$

with probability

$$P^{(\tau)}(N) \xrightarrow{\text{large } N} |\lambda_{\max}|^{2N} (u_{\max}|u_{\max})\langle v_{\max}| \rho_B |v_{\max}\rangle. \quad (11)$$

Notice that the final pure state $|u_{\max}\rangle$ is independent of the choice of the initial state of system B, i.e., any initial (mixed) state shall be driven to the unique pure state $|u_{\max}\rangle$ by repeated measurements performed on the other system A. Since the asymptotic state $|u_{\max}\rangle$ is one of the eigenstates of the operator $V_\phi(\tau)$, we have the possibility of adjusting the interaction strength and the measurement interval and of choosing an appropriate initial state $|\phi\rangle$ for system A so that a desired pure state $|u_{\max}\rangle$ is realized in system B after a large number of measurements on A [as long as the probability $P^{(\tau)}(N)$ does not become meaningfully small]. This discloses another feature of the Zeno phenomenon: the action of the quantum Zeno-like measurements dramatically affects the dynamics of system B of interest.

A few comments are in order. First, the existence of a unique, discrete, and nondegenerate λ_{\max} of the operator $V_\phi(\tau)$, which has been assumed here, is essential for this purification mechanism. Even though this condition is satisfied for some systems with a discrete spectrum as will be shown in the examples below, some definite mathematical criteria for its validity have yet to be clarified. In particular, it remains open if the present purification mechanism can be applied to systems with continuous spectra. Nevertheless at the same time, it would be worth stressing that not a few discrete quantum systems, including two- or three-level systems which play important roles in the field of quantum information and computation, certainly fall into the category of systems with unique, discrete, and nondegenerate λ_{\max} .

Second, it is easy to see that the approach to the final pure state $|u_{\max}\rangle$ is governed by the ratio between the largest and the second largest (in magnitude) eigenvalues of the operator $V_\phi(\tau)$. It is possible that as the number of degrees of freedom increases, the eigenvalues λ_n distribute more closely to each other, which would make the present purification process less effective. Last, even though the measurements are repeated many times, as in the *bona fide* Zeno case, the present scheme does not explicitly rely on the peculiar quadratic behavior of quantum systems at short times. As far as the essential assumption on the spectrum of the operator $V_\phi(\tau)$ is satisfied, there will be no limit on the time interval τ . Notice that the repetition of one and the same quantum measurement is crucial here.

Let us illustrate the above conclusion in a simple but still nontrivial model. We consider two single-mode harmonic oscillators a and b , in interaction in the rotating-wave approximation. The total Hamiltonian reads

$$H = \Omega a^\dagger a + \omega b^\dagger b + ig(a^\dagger b - ab^\dagger), \quad (12)$$

where the frequencies Ω and ω and the coupling constant g are real parameters. The spectrum is discrete. We prepare system A (oscillator a) in some definite pure state $|\phi\rangle$ (typically a number state $|n_a\rangle$ or a coherent state $|\alpha\rangle$) at time $t=0$ and let it evolve under the above Hamiltonian. Then the initial state of system A starts to evolve towards other states owing to the coupling to system B (oscillator b), the initial state of which can be arbitrary. The state of oscillator a is projected onto its initial state $|\phi\rangle$ at each measurement, and the interval between measurements τ is taken small, compared with the typical time scales of the system, e.g., $2\pi/\delta$ in (14) below.

The eigenvalue problem (5) of the relevant operator $V_\phi(\tau)$ is solved exactly in this case. Indeed, since the time-evolution operator $e^{-iH\tau}$ can be factorized as

$$e^{-iH\tau} = e^{Aa^\dagger b} e^{Ba^\dagger a} e^{Cb^\dagger b} e^{-Aab^\dagger}, \quad (13)$$

in terms of the τ -dependent functions

$$A = \frac{(g/\delta) \sin \delta \tau}{\cos \delta \tau + i[(\Omega - \omega)/2\delta] \sin \delta \tau}, \quad (14a)$$

$$B = -\frac{i}{2}(\Omega + \omega)\tau - \ln \left[\cos \delta \tau + i \frac{\Omega - \omega}{2\delta} \sin \delta \tau \right], \quad (14b)$$

$$C = -\frac{i}{2}(\Omega + \omega)\tau + \ln \left[\cos \delta \tau + i \frac{\Omega - \omega}{2\delta} \sin \delta \tau \right], \quad (14c)$$

where $\delta = \sqrt{g^2 + (\Omega - \omega)^2/4}$, we easily find the eigenvectors $|u_n\rangle$ and $|v_n\rangle$ of the operator $V_\phi(\tau)$, once the initial state $|\phi\rangle$ of oscillator a is specified.

If we prepare oscillator a in the number state $|n_a\rangle$ at $t=0$, the relevant operator is calculated to be

$$V_{n_a}(\tau) = \sum_{k=0}^{n_a} \frac{n_a!}{[(n_a - k)!]^2 k!} e^{kB} (-A^2 e^C)^{n_a - k} \times e^{Cb^\dagger b} \prod_{\ell=1}^{n_a - k} (b^\dagger b + \ell), \quad (15)$$

from which we understand that the number states $|n_b\rangle$ of oscillator b constitute the set of eigenvectors of the operator (15). Therefore the state of oscillator b is driven to a number state *irrespective* of its initial state, when the coupled oscillator a is repeatedly confirmed to be in the number state $|n_a\rangle$. The state of system B is *purified* into a number state.

On the other hand, when oscillator a is prepared in a coherent state $|\alpha\rangle$ and found to be in this state at every τ , the relevant operator is rearranged to be

$$V_\alpha(\tau) = e^{-[1 - e^B - A^2/(1 - e^{-C})]|\alpha|^2} e^{D(b^\dagger, b)}, \quad (16)$$

where the operator $D(b^\dagger, b)$ is expressed as

$$D(b^\dagger, b) = C \left[b^\dagger + \frac{A\alpha^*}{1 - e^{-C}} \right] \left[b - \frac{A\alpha}{1 - e^{-C}} \right]. \quad (17)$$

It is easily understood that the state of oscillator b approaches a coherent state $b|\beta\rangle = \beta|\beta\rangle$ with $\beta = A\alpha/(1 - e^{-C})$, since this is the right eigenvector of $D(b^\dagger, b)$ belonging to zero eigenvalue and therefore that of $V_\alpha(\tau)$ belonging to the largest (in magnitude) eigenvalue [14]. The state of system B is again *purified* into (another) pure state $|\beta\rangle$.

In Fig. 1, the survival probability $P^{(\tau)}(N)$ and the so-called fidelity $F^{(\tau)}(N)$

$$F^{(\tau)}(N) = (u_{\max} | \rho_B^{(\tau)}(N) | u_{\max}) / (u_{\max} | u_{\max}) \quad (18)$$

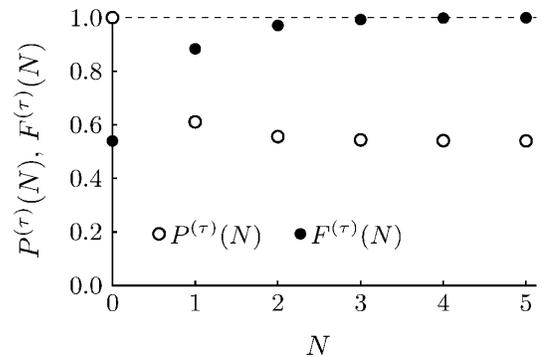


FIG. 1. Probability $P^{(\tau)}(N)$ and fidelity $F^{(\tau)}(N)$ for the model (12) when the initial state of system A, onto which N measurements are performed, is a coherent state $|\alpha\rangle$ and that of system B is $\rho_B \propto e^{-\omega b^\dagger b/k_B T}$ (thermal, i.e., maximally mixed). The parameters are taken to be $\omega = 1$, $g = 0.2$, $T = 1$, $\alpha = 0.5$, and $\tau = 2\pi/[(\Omega + \omega)/2 + \delta] \simeq 5.24$ in units such that $\hbar = k_B = \Omega = 1$. τ is tuned so as to satisfy the condition $|\lambda_{\max}| = 1$ [13], and the ratio of the second largest (in magnitude) eigenvalue to the largest one λ_{\max} is $|e^C| = \sqrt{1 - (g/\delta)^2 \sin^2 \delta \tau} \simeq 0.37$ [14].

are shown as functions of the number of measurements N for the case (16) and (17) with a particular choice of parameters. In order to make the purification procedure more effective, it is preferable that (i) the magnitude of the largest eigenvalue of the operator $V_\phi(\tau)$ be close to 1, $|\lambda_{\max}| \simeq 1$, which maintains the probability $P^{(\tau)}(N)$ large enough even for large N [see (11)], and (ii) the other eigenvalues be all small (in magnitude) compared with $|\lambda_{\max}|$, in order to realize a faster approach to the final pure state $|u_{\max}\rangle$. For this purpose, one may adjust the relevant parameters, such as the interval between measurements τ , the strength of the interaction g , and the state $|\alpha\rangle$ onto which system A is projected. The condition $|\lambda_{\max}| \simeq 1$ is satisfied in general if the interval τ is taken to be small enough as in the ordinary Zeno measurements, but one can optimize this procedure and find better values of τ , not necessarily very small [13], that satisfy both conditions (i) and (ii). See Fig. 1 and its caption, where τ is tuned so as to satisfy the conditions (i) and (ii) for the case (16) and (17), and the purification mechanism becomes very effective after only $N = 2$ steps.

The above arguments clearly and explicitly show how the action of repeated measurements (projections) on one system A can affect the dynamics of the other system B in interaction with A. Interestingly enough, even though the effect of the measurement on the latter system B is indirect, its influence is far reaching if the measurement is repeated many times: irrespectively of its initial (mixed) state, the state of system B is *purified* towards a pure state, provided the conditions on the spectrum of $V_\phi(\tau)$ are satisfied. The final state of system B is prescribed by the total Hamiltonian, the pure state (usually taken to be the initial state) onto which system A is projected by the measurement, and the time interval between successive measurements. This opens a new possibility on how to control the state of a quantum system on which we have no direct access. If another system under our control can be coupled to the former system, we would only have to decide which state has to be measured on the controllable system. After such measurements are performed many times at the prescribed time intervals, the desired pure state would be realized with some probability in the system beyond our control.

Purification of quantum states is now considered to be one of the key technologies for quantum information and computation, and is being widely explored [15] (especially in the context of “entanglement purification”). Compared to some other procedures, the idea here is rather simple: one has only to repeat the same measurements. The objects to which the present method is applicable are general and not restricted to “qubits.” Furthermore, it is worth emphasizing the versatility of this procedure, i.e., the possibility to adjust the target pure state, the balance between fidelity and probability yield, and so on. These issues would deserve further study,

for example, in the context of quantum information and computation.

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