Coherent Regimes of Globally Coupled Dynamical Systems

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This Letter presents a method by which the mean field dynamics of a population of dynamical systems with parameter diversity and global coupling can be described in terms of a few macroscopic degrees of freedom. The method applies to populations of any size and functional form in the region of coherence. It requires linear variation or a narrow distribution for the dispersed parameter. Although an approximation, the method allows us to quantitatively study the transitions among the collective regimes as bifurcations of the effective macroscopic degrees of freedom. To illustrate, the phenomenon of oscillator death and the route to full locking are examined for chaotic oscillators with time scale mismatch.

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Introduction.—Populations of globally coupled dynamical systems represent a useful framework to study the collective properties of biological systems [1], allosterically activated enzymatic reactions [2], electronic devices [3], and chemical reactions [4–6]. Although started as qualitative analyses, in the past years it has appeared that such theoretical approaches can be linked quantitatively to experimental systems, such as electrochemical oscillators [7], arrays of Josephson junctions [8] or metabolic synchronization in suspensions of yeast cells [9]. Globally coupled systems can show different macroscopic behavior when the coupling strength and the parameter dispersion are changed. If the coupling is high enough and the parameter dispersion sufficiently small, the elements of the population evolve in time close to each other (and thus to the mean field) in phase space. In the opposite case, the elements of the population move incoherently and eventually their positions average out, so that the asymptotic dynamics of the mean field is characterized only by fluctuations that vanish in the limit of an infinite number of elements. Between these two limiting cases, complex collective behavior arises. This type of scenario has been observed in a wide number of systems, starting from the early works of Winfree and Kuramoto on phase rotators [10], over limit cycle oscillators [11] to chaotic oscillators [12]. Investigations have been performed with various statistical methods, based on phase reduction [6,13], the continuity equation formalism [14], or the slaving principle [15]. In particular, the introduction of order parameters has appeared to be useful for quantifying the collective regimes, as it aims at providing a direct link between the microscopic and the macroscopic dynamics. However, due to the fact that for low coupling the system spans a region of phase space whose dimension grows with the population size, the onset of macroscopic oscillations from *incoherence* cannot, in general, be described by means of a closed system of a few macroscopic degrees of freedom. In the *coherent* region, on the other hand, the macroscopic dynamics lives on a low dimensional manifold which coincides, in the limit of identical oscillators, with the trajectory of a single uncoupled element of the population, and thus a description in terms of a few effective degrees of freedom may work. Indeed, for limit cycle oscillators with strong coupling and small parameter diversity, an order parameter expansion was recently given by De Monte and d'Ovidio [16], showing that the transient and asymptotic dynamics of the mean field can be accounted for by only two macroscopic variables.

In the present work we propose a method by which the mean field dynamics of globally coupled dynamical systems [maps or ordinary differential equations (ODEs)] with dispersion on one parameter can be systematically reduced (in the coherent regimes) to a system of two coupled order parameters. The conditions we need to impose are either a linear dependence or a narrow distribution for the dispersed parameter and, in general, smooth dynamical systems. The method can be applied to both finite and infinite population sizes.

This Letter is organized in two parts. First, we generalize the method proposed in Ref. [16] to any population of ODEs or maps: the equations of motion of the mean field and of a second order parameter, appearing when the parameter diversity is introduced, are derived from the microscopic equations through a perturbative approach. The theory provides approximate but still quantitative predictions, which are tested on populations of ODEs with time scale mismatch. In particular, we study the phenomenon of *oscillator death,* in which an attracting equilibrium appears in the system due to the interplay between high parameter dispersion and strong coupling. We then show that the approach continues to apply to nonstationary regimes: varying the parameter spread, the transition from oscillator death to full synchronization in

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a population of Rössler systems is followed, and this allows us to characterize the various locked regimes in terms of a cascade of macroscopic bifurcations.

Order parameter expansion.—Let us consider a population of N dynamical systems in the coherence region, i.e., such that the distance $\|X - x_j\|$ between the mean field X and the state x_j of any of the oscillators remains small in time. For ODEs the dynamics of the jth element of the population is defined by the equation

$$\frac{dx_j}{dt} = f(x_j, p_j) + K(X),$$

while for maps

$$x_j \rightarrow f(x_j, p_j) + K(X).$$

To simplify the notation, we write in general $\mathcal{T} x_j = f(x_j, p_j) + K(X)$, where $\mathcal{T} x_j$ is the operator of time evolution applied to x_j. The variables $x_j \in \mathbb{R}^n$ are the state vectors and $p_j \in \mathbb{R}$ is a real parameter taken from a distribution of average $p_0 = \langle p \rangle$ and variance $\sigma^2 = \langle (p - p_0)^2 \rangle$. The oscillators are assumed to be globally coupled to the mean field $X = \langle x \rangle \equiv \sum_{j=1}^{N} x_j/N$ through the function $K:\mathbb{R}^n \rightarrow \mathbb{R}^n$. A change of variables $x_j = X + \epsilon_j$ expresses the position of every element of the population in terms of the mean field, so that in the coherent regime a series expansion in ϵ_j can be performed. Setting $\delta_j = p_j - p_0$ and writing the mixed differentials in x and p as $\mathcal{D}_{x,p}$, we get

$$\mathcal{T} x_j = f(X, p_0) + \mathcal{D}_x f(X, p_0)\epsilon_j + \mathcal{D}_p f(X, p_0)\delta_j + \mathcal{D}_{x,p} f(X, p_0)\epsilon_j \delta_j + o(|\epsilon_j|^2) + C. \quad (1)$$

The terms in $o(|\epsilon_j|^2)$ are small because of the hypothesis of coherence. C contains all the Taylor terms that are not $O(|\epsilon_j|^2)$, i.e., terms with a factor of the type: $\mathcal{D}_{x,p_m} \delta^m$ with $m > 1$. This term may be neglected if the dependence on the parameter is linear or if the parameter distribution is narrow. In fact, in the first case $\mathcal{D}_{x,p_m} \delta^m = 0$, $\forall m > 1$, while in the second the term is $O(\delta_j^2)$. We can thus discard these two terms and construct the macroscopic equations, by averaging Eq. (1) and using $\langle \epsilon \rangle = \langle \delta \rangle = 0$:

$$\mathcal{T} X = f(X, p_0) + \mathcal{D}_{x,p} f(X, p_0)\langle \delta \epsilon \rangle.$$

The only first order term that is left defines a new macroscopic variable $W := \langle \delta\epsilon \rangle$, that we call the *shape parameter.* It measures the dispersion in both the parameter and the phase space.

In order to close the system, the equation of motion for W has to be obtained as well. As for the mean field, this can be done by writing $\mathcal{T} W = \mathcal{T}\langle \delta\epsilon \rangle = \langle \delta \mathcal{T} x \rangle$ and then using Eq. (1). Discarding again higher order terms and making the closure assumption that the term $\langle \epsilon\delta^2 \rangle$ is negligible relative to $\langle \delta^2 \rangle$ (which is true in the coherent regimes), a closed (approximated) system is obtained for the two macroscopic variables:

$$\begin{cases} \mathcal{T} X = f(X, p_0) + \mathcal{D}_{x,p} f(X, p_0) W + K(X) \\ \mathcal{T} W = \sigma^2 \mathcal{D}_p f(X, p_0) + \mathcal{D}_x f(X, p_0) W. \end{cases} \quad (2)$$

The mean field thus behaves like an uncoupled individual element if the oscillators are identical (since in this case $W = 0$). However, parameter diversity may induce new regimes. We remark that the method can be applied in the same way for other choices of the coupling term.

Populations with time scale diversity.—Let us now apply the order parameter reduction to the coherent behavior in populations of oscillators with time scale diversity. These oscillators may be regarded as a generalization of oscillators with different natural frequencies (e.g., the Kuramoto model) and are defined by the equation

$$\frac{dx_j}{dt} = \tau_j g(x_j) + \mathcal{K}(X - x_j). \quad (3)$$

τ_j are strictly positive numbers (which rescale the speed along the orbit) and the matrix $\mathcal{K} = kI$ provides an isotropic diffusive coupling.

Oscillator death.—As a first application, we address the phenomenon of oscillator death, which was first described in limit cycle oscillators with a large natural frequency mismatch and strong coupling [15,17]. Under these conditions, all the elements eventually collapse on the origin, which is an unstable focus for the uncoupled systems. It was then supposed that the phenomenon is more general [18] and it was actually described in many other systems (such as Brusselators [5] and biological systems [19]) where an equilibrium, unstable in the uncoupled case, becomes attracting and suppresses the periodic or chaotic oscillations. When the oscillator death takes place, the regime is trivially coherent since all the elements of the population lie on the same equilibrium (which does not need to be unique) and the dependence on the dispersed parameter is linear, so that the requirements for applying the order parameter reduction are both satisfied. The order parameter expansion allows us to treat the problem in general and with a simple analysis, showing that an unstable equilibrium of g (that, without loss of generality, we put at the origin) can always be stabilized by large time scale dispersion and high coupling, provided that the equilibrium is a saddle focus whose repelling eigenvalues have an imaginary part larger than the real one. In the reduced system Eq. (2), the macroscopic equilibrium corresponding to oscillator death is the point $X = 0$, $W = 0$ and its stability can be studied with simple algebra (the Jacobian matrix is composed by four blocks which can be simultaneously diagonalized). For each eigenvalue $\lambda^l = a_l + ib_l$ of the unstable equilibrium of $g(x)$, two macroscopic eigenvalues appear:

$$\Lambda^l_{1,2} = \lambda_l - \tfrac{k}{2} \pm \sqrt{(\tfrac{k}{2})^2 + \lambda_l^2 \sigma^2}. \quad (4)$$

It is clear that if the equilibrium for the uncoupled system is repelling due to a purely real and positive eigenvalue, then the corresponding Λ always has a positive real part and the oscillator death solution cannot be stable. Instead, if the instability is due to couples of complex conjugates, the condition for the oscillator death to be stable is that the real parts of the eigenvalues (4) are all negative.

Let us now consider the case in which only two eigenvalues of the uncoupled system are complex conjugate, while the others are real and negative. This is the only case where oscillator death can appear in populations of globally coupled two- or three-dimensional systems, such as coupled Lorenz or Rössler oscillators. The bifurcation condition Eq. (4) now reduces to a single inequality. Recalling the definition of λ, the boundary of the oscillator death $\mathrm{Re}(\Lambda) = 0$ (where the indices have been dropped due to the fact that there is only one pair of complex eigenvalues) can be rewritten in the form

$$\left| \mathrm{Re}\left[\sqrt{1 + 4(1 + i\gamma)^2 \left(\frac{\sigma}{k^*} \right)^2} \right] \right| = 1 - \frac{2}{k^*}. \qquad (5)$$

Here $k^* = k/a$ and $\gamma = b/a$, corresponding to a rescaling of the variables so that the real part of the uncoupled system's eigenvalues is unitary. The bifurcation boundary is thus a surface in the space (k^*, σ, γ). The fact that only three parameters account for the stability of oscillator death in any population with the aforementioned characteristics follows from the fact that oscillator death is essentially a local phenomenon, so that the nonlinearities influence at most the critical character of the macroscopic Hopf bifurcation. In the plane (k^*, σ), the bifurcation boundaries are lines parametrized by γ (Fig. 1). Their asymptotic behavior can be computed from Eq. (5) and gives $k^* = (\gamma^2 - 1)\sigma^2$ (upper boundary, to locking) and $k^* = 2(1 + \sigma)$ (lower boundary, to incoherence). Moreover, if the imaginary part of λ is much larger than the real one (i.e., in the limit $\gamma \rightarrow \infty$), a further rescaling is possible: $\sigma^* = \sqrt{\gamma^2 - 1}\sigma$. In this limit, the bifurcation boundaries can be rescaled to the same line: $k^* = \sigma^{*2}$ and $k^* = 2$. Oscillators as different as Rössler and Lorenz systems rescale to the same curve obtained analytically from Eq. (5) (Fig. 2).

Nonsteady coherent regimes.—Let us now speculate about what happens outside the region of oscillator death. On the lower boundary the coupling is weak relative to the parameter spread and the system will bifurcate into incoherence; therefore, we expect the expansion to no longer be valid. On the upper boundary, by contrast, where the closure assumptions are satisfied, the phase transition to the locked state arises macroscopically as a Hopf bifurcation. The validity of the introduced approximations, moreover, is maintained in the locked regime, where the oscillators remain close to the mean field. For this reason, not only the transition out of the oscillator death but *any other bifurcation up to the fully synchronized state can be identified through the reduced system.* As an example, Fig. 3 compares the behavior of the mean field of a population of 32 Rössler oscillators with time scale mismatch to the order parameter expansion Eq. (2). Changing the spread σ in the time scale distribution, a complete period doubling cascade can be followed, connecting the fully locked regime to oscillator death, with remarkable quantitative agreement. From the validity of the order parameter expansion in describing the macroscopic chaotic dynamics, one can infer that the Lyapunov spectrum of the reduced system is composed by the dominant eigenvalues of the full system. The microscopic degrees of freedom which are neglected in the approximation would therefore act as a perturbation on the collective dynamics and give a small contribution to the Kaplan-Yorke dimension of the macroscopic attractor.

Conclusions.—In ensembles of globally coupled oscillators, parameter diversity may induce nontrivial collective behavior, where the mean field dynamics is qualitatively different from that of each uncoupled

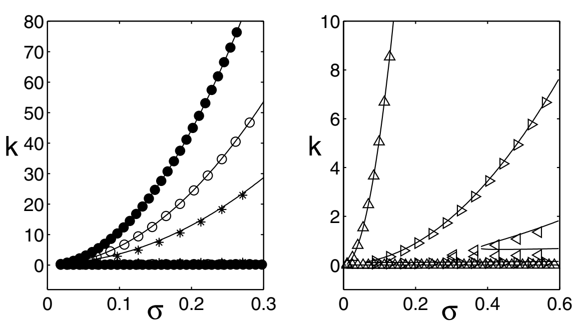

FIG. 1. Comparison of the bifurcation boundaries for the oscillator death in the full and the reduced system for different populations of dynamical systems: Lorenz ($p = 10$, $b = 8/3$) $r = 28$ (●), $r = 32$ (○), $r = 50$ ($*$); Rössler ($b = 0.4$, $c = 8$) $a = 0.01$ (△), $a = 0.1$ (◁), $a = 0.4$ (▷). The continuous lines are the bifurcation boundary for the corresponding reduced systems. Populations of 100 elements and Gaussian parameter distributions have been used.

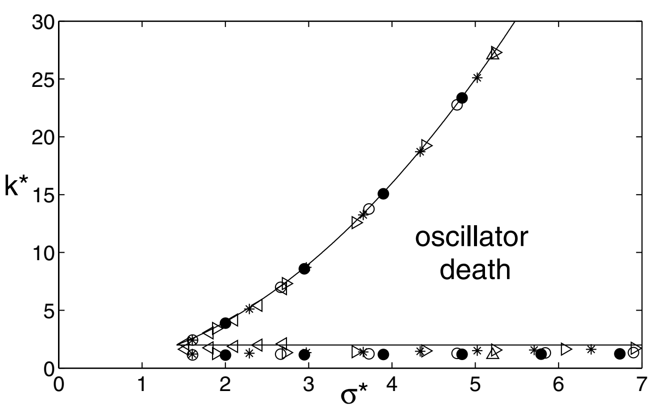

FIG. 2. The bifurcation boundaries of oscillator death for different types of oscillators collapse on the same, analytically determined curve (continuous line) after rescaling the parameters k and σ. Symbols as in Fig. 1.

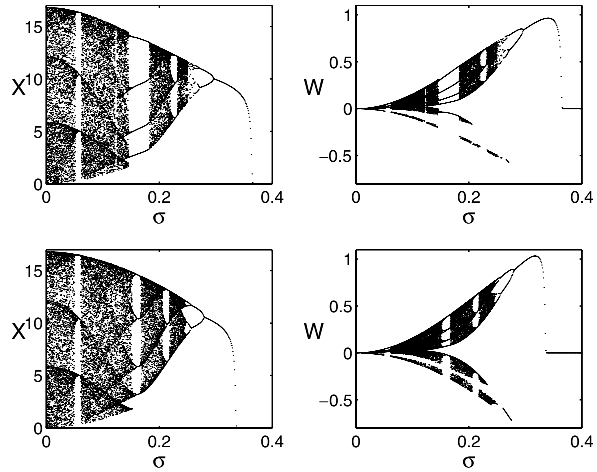

FIG. 3. Poincaré section of the mean field and of the shape parameter for a population of 32 coupled Rössler oscillators ($a = 0.25$, $b = 1$, $c = 8.5$) with time scale mismatch (top) and for its order parameter reduction (bottom). The coupling is $k = 1$. The systems go from chaos to oscillator death when the standard deviation σ of the parameter distribution increases. The reduced system reproduces the bifurcation cascade of the population with remarkable quantitative agreement.

element. In this work we have shown that such regimes are low dimensional in the region of coherence and that a description by means of few effective degrees of freedom may be given. This is done through an expansion around the perfectly synchronized state. Although the method involves some approximations (it is not exact even for a linear analysis), it provides nevertheless an accurate and quantitative description of the dynamics at the macroscopic level. We remark that there are no requirements on the population size: as far as the oscillators are of the same type, populations of different sizes behave the same if they have the same coupling term and the same variance of the parameter distribution. The somewhat surprising consequence of this fact is that the macroscopic features of the coherent regimes can be accounted for by a system of just *two* coupled oscillators, providing a two-body approximation of the population dynamics. Finite size effects, however, do arise as the region of incoherence is approached, and the dimension of the collective dynamics increases. There are several ways in which we think our approach can be developed further. In particular, the inclusion of higher order terms may allow us to explain more complex collective regimes arising close to incoherence. Moreover, a similar approach accounts for the effect of noise on the collective dynamics of identical oscillators. A macroscopic bifurcation scenario similar to that induced by parameter diversity appears in large populations of noisy chaotic maps. This bifurcation and the finite size effects can again be described in the framework of an order parameter expansion [20].

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