Interior Gap Superfluidity

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We propose a new state of matter in which the pairing interactions carve out a gap within the interior of a large Fermi ball, while the exterior surface remains gapless. This defines a system which contains both a superfluid and a normal Fermi liquid simultaneously, with both gapped and gapless quasiparticle excitations. The universality class of this state can be realized at weak coupling. We predict that a cold mixture of two species of fermionic atoms with different mass will exhibit this state. For electrons in appropriate solids, it would define a material that is simultaneously superconducting and metallic.

Recent developments in ultracold alkali atomic gases [1] have revitalized interest in some basic qualitative questions of quantum many-body theory, because they promise to make a wide variety of conceptually interesting parameter regimes, which might previously have seemed academic or excessively special, experimentally accessible. With this motivation, and stimulated by questions in quantum chromodynamics (QCD) at high density [2–4], we here revisit the question of fermion pairing between species whose Fermi surfaces do not precisely match. We have found a possibility that seems to be new and certainly is interesting, and which could turn out to be relevant even for conventional solids.

The standard Bardeen-Cooper-Schrieffer (BCS) [5] theory of superconductivity describes pairing between particles of equal and opposite momentum near a common Fermi surface. For classic s-wave superconductors the pairing occurs between electrons of opposite spin. In the presence of a weak magnetic field, and, in particular, in the case of ferromagnetic order, the Fermi surfaces of the opposite spins will not match, and the Cooper pairing instability, which was enhanced by vanishing energy denominators, will no longer occur at arbitrarily weak coupling. Larkin and Ovchinnikov and independently Fulde and Ferrell [6] showed that in this circumstance it might be favorable to effectively relatively translate the Fermi surfaces, pairing at a nonzero total momentum (LOFF phase).

A simpler situation, conceptually, is that pairing occurs between two species whose Fermi surfaces do not match simply because their densities or effective masses differ. This possibility arises in several contexts. (i) In ultracold atom systems, it could occur simply because there are atoms of different elements. (ii) In solids it could occur for electron populations in two different bands. (iii) In QCD it occurs for different species of quarks (up, down, strange). If the mismatch is small and the two species are alternative states of the same particle (such as the spin up and down states of electrons or two hyperfine-spin states of cold ^{40}K or ^{6}Li atoms as prepared in experiments [7–10]), it can be favorable to equalize the Fermi surfaces, absorbing a cost in kinetic energy, and then to pair at zero momentum following BCS. For larger mismatches, LOFF-type ordering can occur. More elaborate forms of position-dependent ordering, with the superfluid gap having standing wave or even crystalline structure, have been found to be favorable in models of QCD at high density [11].

None of these possibilities, however, extrapolates to what we might expect at strong coupling. Given strong attraction between the species, we would just expect to bind as many paired quasimolecules as possible. At low temperature, these quasimolecules will form a Bose-Einstein condensate. The residual unpaired particles will constitute a separate normal fluid. It is natural to inquire whether there is a weak-coupling phase that matches this behavior qualitatively. We now identify such a phase.

Consider a homogeneous fermion gas in three dimensions containing two species ($\alpha = A, B$) obeying simple parabolic dispersion relations, described by the Hamiltonian

$$H = \sum_{\mathbf{p}\alpha} \epsilon_{\mathbf{p}}^{\alpha} \psi_{\mathbf{p}}^{\alpha\dagger} \psi_{\mathbf{p}}^{\alpha} + g \sum_{\mathbf{pp'q}} \psi_{A\mathbf{q}+\mathbf{p}}^{\dagger} \psi_{B\mathbf{q}-\mathbf{p}}^{\dagger} \psi_{B\mathbf{q}-\mathbf{p'}} \psi_{A\mathbf{q}+\mathbf{p'}}, \quad (1)$$

where $\epsilon_{\alpha}(\mathbf{p}) = \mathbf{p}^2/2m_{\alpha} - \mu_{\alpha}$. See Fig. 1. (More precisely, we assume that this interaction exists so long as the momenta of the particles are within a strip of size λ around the smaller Fermi surface; λ will later serve as an ultraviolet cutoff.) Our heuristic analysis will not distinguish whether or not the species are strictly conserved separately. We define chemical potentials so that the Fermi surfaces for both species are at $\epsilon_F = 0$. We shall be interested in cases where $m_A < m_B$ and $\mu_A > \mu_B$, in such a way that the Fermi momentum for the species B is greater than that for the species A, $p_F^B > p_F^A$.

We suppose that there is an attractive effective interaction in s wave between particles of different species ($g < 0$), and that the coupling is weak, so that we can construct our ground state by modifying the ground state of the noninteracting system. If the Fermi surfaces matched, the attractive interaction would trigger standard
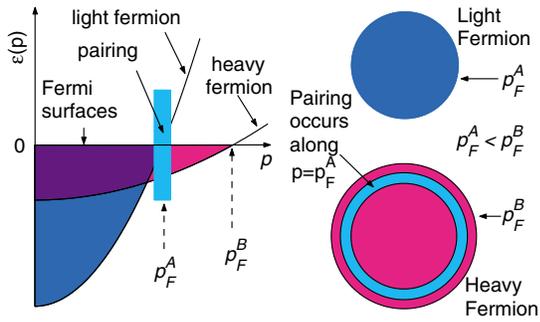
FIG. 1 (color online). The prototype situation where we anticipate formation of an interior gap superfluid at weak coupling. There are two species of fermions with different band structures, here both taken as isotropic and parabolic, but with different effective masses and different sizes in momentum space. When the larger Fermi ball derives from relatively flat band (large effective mass) and the interaction near the momentum surface defined by the smaller Fermi sphere is attractive, it can be favorable to form correlated pairs near this smaller sphere, even at the cost of promoting some particles of the heavier species to the exterior Fermi sphere. One will then have both superfluidity, with a momentum gap at the smaller sphere, and normal Fermi liquid excitations at the larger sphere.

BCS superfluidity, with Cooper pairing of equal and opposite momenta. The BCS wave function, however, postulates either zero or double occupancy of the paired modes, and it is incompatible with keeping the modes of species B between p_F^A and p_F^B completely filled. If we are to support pairing of total momentum zero we must promote some particles of species B up to momenta near p_F^B, thus carving an interior "trench" of the species B Fermi sea near momentum $p = p_F^A$.

There is competition between the energetic cost of such promotions and the gain from pair formation, and it may not be obvious whether there can be a net profit in any nontrivial case. To assess this, let us suppose that the pairing introduces a *momentum* gap of order κ. By this we mean that in an interval of order κ around p_F^A we will take superpositions of unoccupied and doubly occupied states, as in ordinary BCS theory. In particular, we do not automatically fill the single-particle states for species B, even though they are below the free-particle Fermi momentum p_F^B. One could also, more awkwardly but perhaps more properly, speak of normalized energy gaps of order $\kappa p_F^A/m_\alpha$ for the two species. However phrased, the point is that it is important to prepare an equal number of modes to pair, and state counting takes place in momentum space. The condensation energy must be of the same energy as the spectral displacement of the particles, so we have for the energy gain ϵ_{pair} per pair $\epsilon_{\mathrm{pair}} \sim p_F^A \kappa/\tilde{m}$, with $\tilde{m} \equiv m_A m_B/(m_A + m_B)$ the reduced mass. On the other hand, the density of pairs n_{pair} is of the order of $n_{\mathrm{pair}} \sim 4\pi p_F^{A2} \kappa$. In order to accommodate the depletion of species B, which is of the same order as the number of pairs, we must promote a corresponding number of par-

ticles from p_F^A to p_F^B, which costs $(1/2m_B)(p_F^{B2} - p_F^{A2})$ per pair. Putting it all together, to make a net profit we require $\epsilon_{\mathrm{pair}} \sim p_F^A \kappa/\tilde{m} > (p_F^{B2} - p_F^{A2})/2m_B$.

The region of pairing in momentum space will be strictly in the interior of the larger Fermi surface — clearly distinct from its boundary — and we will realize interior gap superfluidity, if $p_F^B - p_F^A > \kappa$. This is compatible with our earlier condition for

$$1 > \frac{p_F^B + p_F^A}{2p_F^A} \frac{m_A}{m_A + m_B}. \tag{2}$$

This consistency condition can be satisfied, specifically for $m_B \gg m_A$. Note particularly that it is independent of the gap κ. Looking back to our net profit condition, we see that κ can be taken arbitrarily small, for sufficiently large m_B/m_A. Thus interior gap superfluidity can take place at weak coupling, where the mean-field assumptions implicit in this heuristic analysis are valid. In the class of models under discussion, therefore, we find a robust weak-coupling phase characterized by a (momentum) gapped Fermi surface interior to a surface with unpaired excitations. For charged fermions, it is a phase that is simultaneously superconducting and metallic at zero temperature.

To construct the interior gap ground state explicitly we generalize the standard BCS wave function as follows

$$|\Psi_{\mathrm{IG}}\rangle = \prod_{|\mathbf{p}| \le p_\Delta} (\sin\theta_{\mathbf{p}} + \cos\theta_{\mathbf{p}} \psi_{A\mathbf{p}}^\dagger \psi_{B,-\mathbf{p}}^\dagger) \prod_{|\mathbf{p}| > p_\Delta} \psi_{B\mathbf{p}}^\dagger |0\rangle,$$

where the $\theta_{\mathbf{p}}$'s *and* p_Δ are variational parameters. As usual, there is a manifold of degenerate states featuring an overall relative phase between the $\sin\theta_{\mathbf{p}}$ and $\cos\theta_{\mathbf{p}}$ factors. The order parameter is of usual form: $\langle \psi_{A\mathbf{p}}^\dagger \psi_{B,-\mathbf{p}}^\dagger \rangle_{\mathrm{IG}} = \sin\theta_{\mathbf{p}} \cos\theta_{\mathbf{p}}$. Upon variation with respect to $\theta_{\mathbf{p}}$ and p_Δ, we find $\cos^2\theta_{\mathbf{p}} = \frac{1}{2}\{1 - [\epsilon_{\mathbf{p}}^+/\sqrt{(\epsilon_{\mathbf{p}}^+)^2 + \Delta^2}]\}$ and $p_\Delta^2 = \frac{1}{2}(p_F^{B2} + p_F^{A2}) - \frac{1}{2}[-16\Delta^2 m_A m_B + (p_F^{B2} - p_F^{A2})^2]^{1/2}$, with $\epsilon_{\mathbf{p}}^+ \equiv \frac{1}{2}(\epsilon_{\mathbf{p}}^A + \epsilon_{\mathbf{p}}^B)$. The gap parameter, defined here as $\Delta = -g \sum_{|\mathbf{p}| \le p_\Delta} \langle \psi_{A\mathbf{p}}^\dagger \psi_{B,-\mathbf{p}}^\dagger \rangle_{\mathrm{IG}}$, satisfies the integral equation $1 = -g \sum_{|\mathbf{p}| \le p_\Delta} (1/\sqrt{\epsilon_{\mathbf{p}}^{+2} + \Delta^2})$.

An important departure from standard BCS theory occurs because the energy difference between paired and unpaired modes no longer becomes arbitrarily small. For this reason one is not doing degenerate perturbation theory, and does not encounter a true infrared divergence (vanishing energy denominators) in the Cooper pairing channel. As a consequence, the gap equation supports a nonzero solution only for $|g| > g_c$, with

$$g_c \simeq 2\left[N_+(0) \ln\left(\frac{p_0 \lambda}{p_F^{B2} - p_F^{A2}} \frac{m_A + m_B}{m_A} \right) \right]^{-1}, \tag{3}$$

where we have introduced the generalized density of states $N_+(0) \equiv \sum_{\mathbf{p}} \delta(\epsilon_{\mathbf{p}}^+)$ and p_0 is the point where $\epsilon^+(\mathbf{p}_0) = 0$. Note, however, that $g_c \to 0$ when $m_B/m_A \to \infty$ for fixed $p_F^{A,B}$, so that interior gap superfluidity can be favorable for arbitrarily weak attractive

interactions, as we anticipated. (Since the dependence on m_B/m_A is logarithmic, in practice it may be difficult to realize the interior gap state at extremely weak coupling. See Fig. 2 below.) Numerically, we typically find that g_c is rather smaller.

It is straightforward to calculate the condensation energy. Up to terms of higher order in Δ, we find

$$E_{\mathrm{IG}} - E_{\mathrm{N}} = -\frac{1}{2} N_+(0) \Delta^2 \left[\frac{1}{2} + x_\Delta e^{\mathrm{arcsinh}\, x_\Delta} \right], \quad (4)$$

where $x_\Delta = \epsilon^+(\mathbf{p}_\Delta)/2\Delta$ and E_N is the normal state energy at $\Delta = 0$. For weak pairing p_Δ is close to p_F^A, in which case $\epsilon^+(p_\Delta)$ is negative and finite. In the limit $\Delta \to 0$, x_Δ approaches minus infinity, and the two terms in the brackets cancel. One sees that, for a given gap parameter Δ, the interior gap state gains less condensation energy than a conventional BCS state.

Thus we have demonstrated that the normal state is unstable against formation of an interior gap superfluid when the attractive coupling is strong enough, i.e., $|g| > g_c$. The LOFF state is another candidate for pairing of mismatched Fermi surfaces. We have calculated the ground-state energy difference between these two candidate states numerically. Figure 2 shows a typical phase diagram. We have considered here only the simplest LOFF state, with pairing order parameter $\propto \Delta_{\mathrm{LOFF}} e^{i\mathbf{Q}\cdot\mathbf{x}}$ with $|\mathbf{Q}|$ of order $p_F^B - p_F^A$ [6,12].

A few remarks concerning this phase diagram are in order. The critical coupling constant g_c in Eq. (3) in principle governs the (second-order) phase transition from the normal state to the interior gap state, rather than that from the LOFF state. However, we find in our numerical calculation that the phase transition line between the interior gap and LOF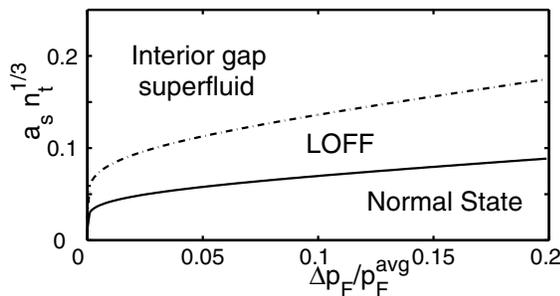F states roughly coincides with the onset of the interior gap ordering. Indeed, since an LOFF particle pair carries a total momentum $\mathbf{Q} \neq 0$, the pairing process in the LOFF state occurs mostly within the intersection of two closed shells of thickness κ, one centered on $\mathbf{Q}$ and the other on zero. The intersection of these shells is a closed ring of thickness κ. By contrast, the interior gap state pairing occurs within a full two-dimensional shell. Since the density of states involved in pairing for the interior gap state is larger than that for the LOFF state, the former develops an order parameter which increases (as a function of intrinsic coupling strength) exponentially faster than the latter, and rapidly dominates once it sets in. On the other hand, the LOFF phase can be realized also for $m_A > m_B$, that is when the heavier fermion has the smaller Fermi surface, when the interior gap phase is not available. This is the case of primary interest for possible phases of quark matter in neutron star interiors, where the B species is the strange quarks. BCS states correspond to the region in our phase diagram where the two species have approximately equal Fermi momentum, $\Delta p_F \sim 0$.

The average particle occupation number $n_\mathbf{p}^{A,B}$ has an unconventional form. $n_\mathbf{p}^A = n_\mathbf{p}^B = \cos^2\theta_\mathbf{p}$ for $|\mathbf{p}| \leq p_\Delta$, and $n_\mathbf{p}^A = 0$ and $n_\mathbf{p}^B = 1$ for $p_\Delta < |\mathbf{p}| \leq \tilde{p}_F^B$, where $\tilde{p}_F^B$ is the shifted Fermi momentum of species B particle due to pairing interaction: $\tilde{p}_F^B \simeq p_F^B(1 + p_0 \tilde{m} \Delta e^{\mathrm{arcsinh} x_\Delta}/p_F^{B3})$. Figure 3 shows the average particle occupation number as a function of momentum at zero temperature. In the interior gap state, the pairing correlation smears the species A Fermi surface slightly — a fraction of single-particle states are depleted below p_F^A and inserted back between
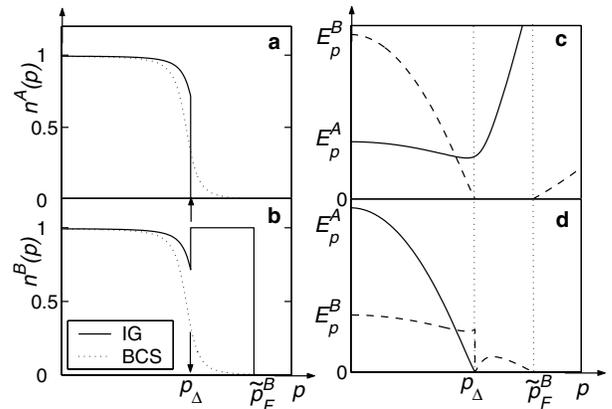

FIG. 2. A typical phase diagram, with the mass ratio $m_B/m_A = 7$. Our coupling constant is $g = -4\pi\hbar^2 a_s/\tilde{m}$, where a_s is the scattering length for s-wave scattering between species A and B, and $n_t = n^A + n^B$ is the total density of particles, calculated using the formula $p_F^\alpha = (6\pi^2 n^\alpha)^{-1/3}$ as for a free Fermi gas. $\Delta p_F \equiv p_F^B - p_F^A$ and $p_F^{\mathrm{avg}} \equiv \frac{1}{2}(p_F^B + p_F^A)$. The dash-dotted line is plotted using the approximate analytical formula (3), while the solid line is plotted by adapting Eq. (3.12) of Takada and Izuyama [12] to the case at hand. Were it plotted directly from our numerical simulations, the phase transition line between interior gap and LOFF would be slightly lowered.

FIG. 3. (a),(b): A plot of average particle occupation number $n_\mathbf{p}^A$ and $n_\mathbf{p}^B$, respectively, in the interior gap superfluid state in comparison with those in a conventional BCS state. For the comparison, we assume the same gap parameter and a matched Fermi surface at $p = p_F^A$. (c) Quasiparticle energy spectra to add a species A or B particle to the system. Corresponding quasiparticle states are $\psi_{A\mathbf{p}}^\dagger|\Psi_{\mathrm{IG}}\rangle$ (solid line) and $\psi_{B\mathbf{p}}^\dagger|\Psi_{\mathrm{IG}}\rangle$ (dashed line). (d) Quasiparticle energy spectra to remove a species A or B particle from the system. Corresponding quasiparticle states are $\psi_{A\mathbf{p}}|\Psi_{\mathrm{IG}}\rangle$ (solid line) and $\psi_{B\mathbf{p}}|\Psi_{\mathrm{IG}}\rangle$ (dashed line).

the normal state Fermi surface at p_F^A and the maximum pairing surface at p_Δ. This distribution does not differ qualitatively from what one finds in a conventional BCS superconducting state. Species B displays a more dramatic contrast. Some modes that are occupied in the normal state, in the Fermi ball interior around $p = p_F^A$, are now depleted, and the deficit is made up at the top of the Fermi surface. This enlarges the Fermi surface from p_F^B to $\tilde{p}_F^B$. Discontinuities in the distribution $n_{\mathbf{p}}^B$ occur at both p_Δ and $\tilde{p}_F^B$.

Approximate quasiparticle excitations of the interior gap state are obtained by diagonalizing the Hamiltonian (1) at mean-field level using the Bogoliubov transformation $\gamma_{1\mathbf{p}}^\dagger = \sin\theta_{\mathbf{p}}\psi_{A\mathbf{p}}^\dagger - \cos\theta_{\mathbf{p}}\psi_{B,-\mathbf{p}}$ and $\gamma_{2,-\mathbf{p}}^\dagger = \sin\theta_{\mathbf{p}}\psi_{B,-\mathbf{p}} + \cos\theta_{\mathbf{p}}\psi_{A\mathbf{p}}^\dagger$. These operators create quasiparticle excitations $\gamma_{1,2;\mathbf{p}}^\dagger|\Psi_{IG}\rangle$ from the interior gap superfluid state. Their spectra are given by

$$E_{1,2}(\mathbf{p}) = \frac{1}{2}(\epsilon_{\mathbf{p}}^A - \epsilon_{\mathbf{p}}^B) \pm \sqrt{\epsilon_{\mathbf{p}}^{+2} + \Delta^2}, \qquad (5)$$

with all energies measured from the shifted Fermi surfaces defined by $\tilde{p}_F^A \simeq p_F^A$ and $\tilde{p}_F^B$. Unlike in conventional BCS theory, there are two branches of excitations: $E_1(\mathbf{p})$ is gapped while $E_2(\mathbf{p})$ is gapless at both $|\mathbf{p}| = p_\Delta$ and $|\mathbf{p}| = \tilde{p}_B^F$. Notably, "$-E_{2\mathbf{p}}$" becomes negative for momenta $|\mathbf{p}|$ between p_Δ and $\tilde{p}_F^B$. This is interpreted consistently as indicating that all such states are filled by species-B particles (note $n_{\mathbf{p}}^A = 0$), and that the absolute value $|E_{2\mathbf{p}}|$ is the energy to create a hole excitation there (Fig. 3).

Given the explicit form of the quasiparticles and their spectrum, phenomenological consequences can be derived along standard lines. The novelty of the interior gap state is that a large manifold of low-energy "normal state" excitations coexists with superfluidity. This spectrum could be probed directly in tunneling experiments. At finite temperature the normal state excitations will be excited, and the appropriate description will involve a two-fluid model incorporating dissipation. In these regards there is some resemblance to conventional superfluids whose order parameter has nodes, such as the p-wave superfluid in liquid ^{3}He, or the d-wave superconducting cuprates. But these states differ from interior gap superfluids both quantitatively, in that the density of gapless modes is much smaller, and qualitatively, in that they involve breaking of rotational symmetry. Another partial analog is the Abrikosov-Gor'kov gapless superconductivity with magnetic impurities [13]; but of course here we do have a gap, and impurities are not a central issue.

Interior gap superfluidity will be realized in a two-species mixture of fermionic cold atoms with different mass. Recent theoretical [14–19] and experimental [7–10] efforts point to the possibility of superfluidity in two-state mixtures of ^{6}Li or ^{40}K atoms. A stable mixture with

different mass could be realized, for instance, in the ^{6}Li and ^{40}K atomic gas [20]. Despite its qualitative difference from BCS-type states, the interior gap state is estimated to have a superfluid transition temperature of the same order as that of Refs. [14–19], for not too weak coupling.

Also, we perceive no problem of principle forbidding the realization of interior gap superfluidity in electron gases, where the species are electrons from different bands, which can have markedly different effective masses. The case of electrons coming from two bands differs, however, from the atomic case in that one should specify a single density and an energy offset between band minima, instead of two independent densities (or chemical potentials). There can be phase transitions as a function of the single density, for example, between interior gap and conventional BCS-type order. Also, of course, the dispersion relations can be different from simple parabolas, which has interesting consequences. We shall return to these questions in a future publication.

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