

Origin of the Universal Roughness Exponent of Brittle Fracture Surfaces: Stress-Weighted Percolation in the Damage Zone

Alex Hansen* and Jean Schmittbuhl†

International Center for Condensed Matter Physics, Universidade de Brasília, 70919-970 Brasília, Distrito Federal, Brazil

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We suggest that the observed large-scale universal roughness of brittle fracture surfaces is due to the fracture propagation being a damage coalescence process described by a stress-weighted percolation phenomenon in a self-generated quadratic damage gradient. We use the quasistatic 2D fuse model as a paradigm of a mode I fracture model. We measure for this model, which exhibits a correlated percolation process, the correlation length exponent $\nu \approx 1.35$ and conjecture it to be equal to that of classical percolation, $4/3$. We then show that the roughness exponent in the 2D fuse model is $\zeta = 2\nu/(1 + 2\nu) = 8/11$. This is in accordance with the numerical value $\zeta = 0.75$. Using the value for 3D percolation, $\nu = 0.88$, we predict the roughness exponent in the 3D fuse model to be $\zeta = 0.64$, in close agreement with the previously published value of 0.62 ± 0.05 . We furthermore predict $\zeta = 4/5$ for 3D brittle fractures, based on a recent calculation giving $\nu = 2$. This is in full accordance with the value $\zeta = 0.80$ found experimentally.

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Fracture surfaces in brittle materials show surprising scaling properties [1]. These were first seen in the mid-1980s [2]. They manifest themselves through self-affine long-range height correlations. That is, the conditional probability density $p(x, y)$, i.e., the probability that the crack surface passes within dy of the height y at position x when it had height zero at $x = 0$, shows the invariance

$$\lambda^\zeta p(\lambda x, \lambda^\zeta y) = p(x, y), \quad (1)$$

where ζ is the roughness exponent. In the early 1990s increasing experimental evidence hinted that the roughness exponent not only existed but had a *universal* value of about 0.80 [3]. The experimental picture today is even more complex: (a) A second, smaller roughness exponent, approximately equal to 0.5, has been observed on small length scales, with a crossover length between the two regimes [4]; (b) the growth of the roughness from an initial straight notch shows anisotropy [5] and a two-regime process in the case of quasibrittle material such as wood [6]; (c) materials such as sandstone, for which the fracture is strongly intergranular, show only a $\zeta = 0.5$ self-affine scaling [7]. Simultaneously with these experiments, theoretical and numerical work have been produced at a steady rate with the aim of (i) understanding why there is a self-affine scaling of the roughness, (ii) why there should be universality of the roughness exponent, and (iii) how to unify observations and modeling [8–12].

It is the aim of this Letter to present a new possible explanation for the observed universal roughness of brittle fracture surfaces at larger scales for mode I cracks. We present our ideas using a paradigm of the fracture model: the quasistatic fuse model [13]. Dynamical fuse models have been proposed and studied in the work of Sornette and Vanneste [14]. The quasistatic fuse model

consists of a lattice where each bond is an Ohmic resistor as long as the electrical current it carries is below a threshold value. If the threshold is passed, the bond burns out irreversibly. The threshold t of each bond is drawn from an uncorrelated distribution $p(t)$. The lattice is placed between electrical bus bars and an increasing current is passed through it. Numerically, the Kirchhoff equations are solved with a voltage difference between the bus bars set to unity. The ratio between current i_j and threshold t_j for each bond j is calculated and the bond having the largest value, $\max_j(i_j/t_j)$, is identified and subsequently irreversibly removed.

In the limit of infinite disorder [i.e., when the threshold distribution is on the verge of becoming non-normalizable, e.g., $p(t) \propto t^{-\alpha-1}$, where $1 \leq t < \infty$ in the limit of $\alpha \rightarrow 0$] the fuse problem becomes equivalent to a bond percolation problem [15]. At more narrow disorders, a rich phase diagram appears which is controlled by two parameters, the exponent α which controls the threshold distribution tail toward infinitely large threshold values and the exponent β which controls the tail of the threshold distribution toward zero: $p(t) \propto t^{-1+\beta}$, where $0 \leq t \leq 1$ [16]. For smaller values of either α or β , the fuse model still shows behavior very similar to percolation: The lattice stops conducting after a finite percentage of bonds have burned out even when the lattice size is extrapolated to infinity. Close to breakdown, critical exponents may be defined precisely as in the percolation problem. However, as the breakdown process in the fuse model is highly correlated, there is no reason to expect these exponents to be equal to those found in the percolation problem. At even smaller disorders, localization sets in.

When the disorder is broad enough so that the fuse model behaves in a percolationlike manner, there is a diverging correlation length $\xi \propto |p - p_c|^{-\nu}$, where p is the density of broken bonds and p_c is the density at which

an infinite lattice breaks down. For classical percolation, $\nu = 4/3$ [17]. For the fuse model away from the infinite-disorder limit, ν has not been measured. Three scenarios are possible for the value of ν : (i) ν depends on the disorder. Hence, it is not a universal quantity. (ii) ν is independent of the disorder but is different from $4/3$. In this case, the fuse model defines a new universality class different from standard percolation. (iii) ν is the same in the fuse model as in standard percolation. Thus, the fuse model is in the universality class of percolation.

In order to determine which of these three scenarios is correct for the two-dimensional fuse model, we studied the survival probability of lattices for different system sizes and different disorders. In Fig. 1, we show survival probability for the threshold distribution $p(t) \propto t^{-1+\beta}$ when $0 \leq t \leq 1$, where $\beta = 1/10$ as a function of density of broken bonds for different lattice sizes. The collapse of the curves obtained for different sizes shows both that the survival probability is converging on a step function at a finite $p = p_c$ and that an estimate of the coefficient $1/\nu = 0.75$. Indeed, we expect the survival probability to scale as $L^{-1/\nu}$. In Fig. 2, we confirm the estimate of ν by showing the 50% survival probability, as a function of lattice size for this threshold distribution and for the threshold distribution $p(t) \propto t^{-1+\beta}$ on the unit interval, where $\beta = 1/3$. Finite size scaling dictates that the effective density at which 50% of the lattices survive, p_s , behaves as

$$p_s = p_c - \frac{c}{L^{1/\nu}}, \quad (2)$$

where c is a constant. By adjusting the value of ν until a straight line ensues, we determine the value of ν . We find

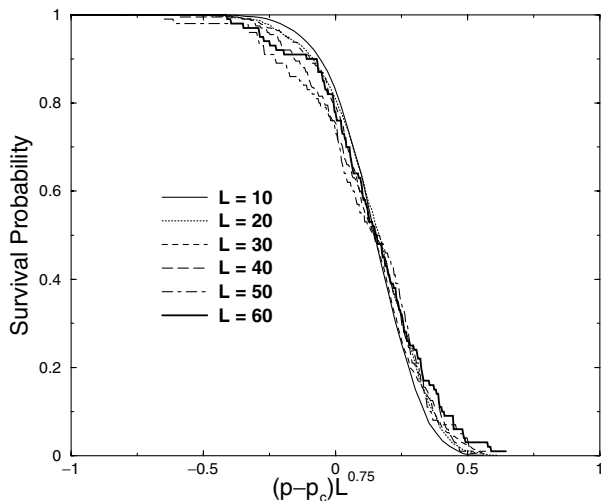


FIG. 1. Survival probability as a function of density of broken bonds plotted against $(p - p_c)L^{1/\nu}$, where $p_c = 0.3735$ and $\nu = 4/3$ gives a good data collapse for different lattice sizes. The threshold distribution was $p(t) \propto t^{-1+\beta}$ on the unit interval where $\beta = 1/10$. The number of samples for each lattice size varied from 2000 for $L = 10$ to 80 for $L = 60$.

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$\nu = 4/3$ fits the data very well. These results are consistent with scenario (3) above: The two-dimensional fuse model is in the same universality class as classical two-dimensional percolation.

If one tries to determine the scaling properties of the final crack for disorders that are so broad that the system behaves as regular percolation, the roughness exponent will be one, since the final crack will be fractal with no anisotropy and so the width of the crack will essentially be that of the lattice itself. However, with more narrow threshold distributions, a nonlinear gradient develops in the damage profile in the average current direction. That is, if y is the average current direction (which is along the voltage gradient or imposed displacement), then damage density averaged in the orthogonal x direction (parallel to the bus bars and along the average final rupture), $\langle p \rangle(y)$, takes the form

$$\langle p \rangle(y) = p_f - A \left(\frac{y - y_c}{l_y} \right)^2, \quad (3)$$

where A is a positive constant that depends on the width of the threshold distribution and l_y is the width of the damage distribution. The damage profile must surely be quadratic as the system must be statistically mirror symmetric about y_c , where the maximum damage occurs. At breakdown, the maximum damage p_f is equal to the critical damage density p_c and can be expressed in terms of the correlation length ξ : $(\langle p \rangle - p_c) \propto \xi^{-1/\nu}$. As proposed by Sapoval *et al.* for percolation in a gradient [18], we suggest to consider the region along the damage zone that is at a distance corresponding to the correlation length: $(y - y_c) \propto \xi$. A point at a distance $y = y_c$ has a finite probability to be within the critical region and so to belong to the final crack. Accordingly, the crack

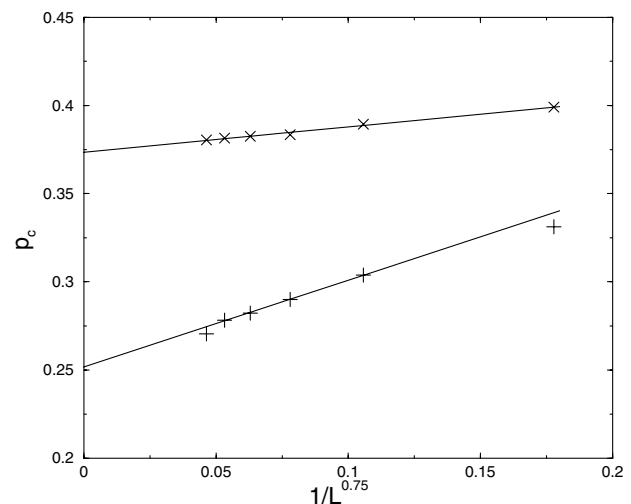


FIG. 2. 50% survival probability plotted against inverse lattice size to the power 0.75 for threshold distributions on the unit interval, $p(t) \propto t^{-1+\beta}$, where $\beta = 1/10$ (\times) and $\beta = 1/3$ ($+$). The straight lines extrapolate to $p_c = 0.3735$ and $p_c = 0.252$, respectively.

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roughness is proportional to the correlation length: $\xi \approx w$ and solving Eq. (3) with the above conditions yield

$$w \sim l_y^{2\nu/(1+2\nu)}. \tag{4}$$

The width of the damage profile, l_y , must be proportional to the width of the system in the direction perpendicular to the final crack, L . The reason for this is that each broken bond creates a disturbance in the average current field that enhances the probability for a new bond to break in a finite-width cone which stretches out from each side of the bond in the direction approximately orthogonal to the average current direction. Hence, as long as the current enhancement is not sufficient to induce crack coalescence and create an unstable crack tip, the damage zone will spread in the new cones in a random fashion. This leads to

$$l_y \propto L. \tag{5}$$

In Fig. 3, we show the damage profile averaged over many samples and for many lattice sizes plotted against y/L . We note from Fig. 3 that the profiles clearly follow Eq. (3). The collapse of the damage profiles shows that they are functions of the combination y/L and L does not enter in any other way. This result is confirmed in Fig. 4 where the width of the damage zone l_y is plotted versus the system size L for two different threshold distributions. Both show a good linear behavior in accordance with Eq. (5). Hence, the width of the crack scales as

$$w \sim L^{2\nu/(1+2\nu)}. \tag{6}$$

We therefore conclude that the fracture roughness exponent is

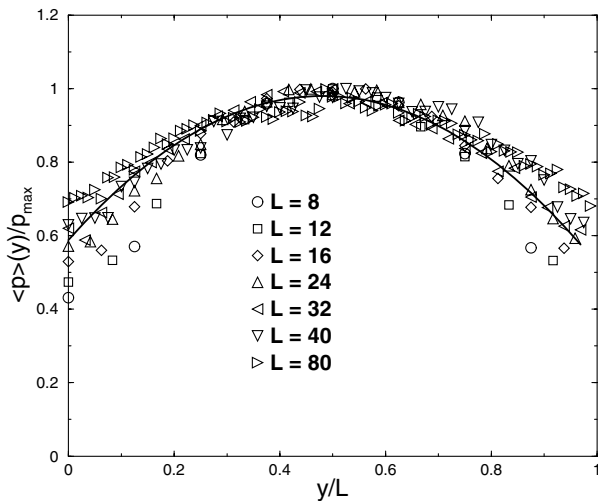


FIG. 3. Damage profile $\langle p \rangle(y)$ normalized so that its maximum is set to unity plotted against y/L for different lattice sizes L and for the threshold distribution $p(t) \propto t^{-1+\beta}$ on the unit interval and where $\beta = 1$. The curve is a quadratic best fit based on the $L = 32$ data in accordance with Eq. (3).

$$\zeta = \frac{2\nu}{1 + 2\nu}, \tag{7}$$

which lead for the 2D fuse model to $\zeta = \frac{8}{11} \approx 0.73$ where we have assumed that $\nu = 4/3$, the standard percolation value. In [8,12], ζ was measured to be about 0.75 in the two-dimensional fuse model. Hence, there is very good agreement with Eq. (7).

Equation (7) is also valid for the 3D fuse model. The roughness of crack surfaces for the 3D fuse model has been characterized numerically by Batrouni and Hansen [10], finding $\zeta = 0.62 \pm 0.05$. On the other hand, Räsänen *et al.* [11] argue, using numerical calculations as support, that the roughness exponent is that of the minimal surface problem, $\zeta = 0.41 \pm 0.02$ [19]. If we assume that the 3D fuse model belongs to the same universality class as classical percolation, we expect $\nu = 0.88$. Using Eq. (7), we obtain $\zeta = 0.64$, which is in very good agreement with the numerical result of Batrouni and Hansen [10].

We now extend our argument to the general case of real 3D brittle fractures in heterogeneous materials. No measurements of ν exist for the brittle fracture problem, neither numerically nor experimentally. However, a recent theory proposed by Toussaint and Pride [20] gives $\nu = 2$, making ν very different from the value found in standard 3D percolation, $\nu = 0.88$ —which is expected for elastic percolation. Using Eq. (7), we arrive at

$$\zeta = \frac{4}{5}, \tag{8}$$

which is indeed in excellent agreement with the experimentally observed roughness exponent for large scales, $\zeta = 0.80$. Hence, a direct measurement of ν in this system is desirable.

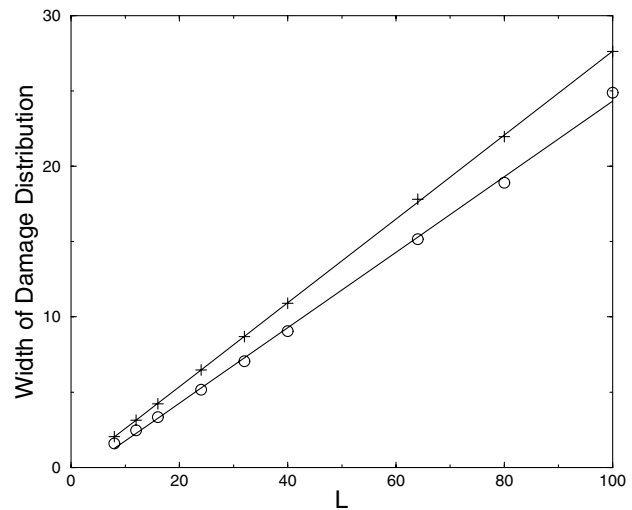


FIG. 4. Width of the damage distributions shown in Fig. 3 (+) and one based on the threshold distribution $p(t) \propto t^{-1+\beta}$ on the unit interval, with $\beta = 1/3$ (O) plotted against L . The straight lines are linear fits to the data.

There have also been experimental [21] and numerical [22] studies of two-dimensional brittle fracture. Roughness exponents between 0.63 and 0.72 and 0.68 ± 0.04 were found in the two experimental studies, and the numerical study gave $\zeta = 0.71 \pm 0.10$. Moukarzel and Duxbury [23] have measured $\nu = 1.16 \pm 0.03$ for two-dimensional elastic percolation. Given that elastic percolation plays the same role in brittle fracture—which is an elastic problem—as ordinary percolation plays in the fuse model, we find using Eq. (7), that $\zeta = 0.70$. This is consistent with both the experimental and numerical values for ζ .

Why should this theory be applicable only to the large-scale exponent observed in brittle fracture, and not the exponent seen at small scales? It is the large-scale exponent that describes the correlated behavior of the damage field which finally will lead to the large-scale properties of fracture surface. The smaller exponent, which is close to 0.5, describes the opening of small cracks and may be caused by corrugation waves propagating elastically along the crack front [24]. At larger scales, where a roughness exponent equal to 0.8 is observed, these waves are too weak to influence the system. At this larger scale, we propose that it is a correlated gradient percolation process which is responsible for the value of the roughness exponent.

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*Permanent address: Department of Physics, NTNU, N-7491 Trondheim, Norway.

Electronic address: Alex.Hansen@phys.ntnu.no

†Permanent address: Laboratoire de Géologie, UMR CNRS 8538, Ecole Normale Supérieure, 24, rue Lhomond, F-75231 Paris Cédex 05, France.

Electronic address: Jean.Schmittbuhl@ens.fr

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