Robust Dynamical Decoupling of Quantum Systems with Bounded Controls

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We propose a general procedure for implementing dynamical decoupling of quantum systems without requiring arbitrarily strong, impulsive control actions. This is accomplished by designing continuous decoupling propagators according to Eulerian paths in the decoupling group for the system. Such Eulerian decoupling schemes offer two important advantages over their impulsive counterparts: they are able to enforce the same dynamical symmetrization but with more realistic control resources and, at the same time, they are intrinsically tolerant against a large class of systematic implementation errors.

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Dynamical decoupling provides a well-defined framework for the manipulation of open quantum systems and interacting quantum subsystems. Inspired by coherent averaging methods in pulsed magnetic resonance spectroscopy [1], and cast in control-theoretic terms in [2,3], decoupling techniques are attracting growing interest from the quantum control and quantum information processing (QIP) communities. In the context of reliable QIP, decoupling is instrumental in the development of quantum error suppression and symmetrization schemes [3–5], with the potential for noise-tolerant universal quantum computation on dynamically generated noiseless subsystems [6]. Variants of the basic decoupling concepts also play a role in protocols for universal quantum simulation of both closed- [7] and open-system [8] dynamics, with implications for encoded simulation [9,10]. In a broader setting, applications of dynamical decoupling to problems that range from suppressing magnetic state decoherence [11] to inhibiting the decay of unstable states [12] or reducing heating effects in linear ion traps [13] have been recently envisaged.

The design of dynamical decoupling schemes has essentially relied so far on the ability to enforce sequences of arbitrarily strong, instantaneous control pulses. That is, it required the ability to impulsively apply a set of control Hamiltonians with unbounded strength [the "bang-bang" (b.b.) assumption [2]]. While providing a convenient starting point, this view suffers from being extremely unrealistic for implementation. In a physical control scenario, additional disadvantages of b.b. decoupling include the difficulty of simultaneously describing the evolution under the natural (drift) Hamiltonian and the control terms, as well as the poor spectral selectivity of b.b. pulses, with substantial off-resonance effects. Finally, although compensation techniques based on composite rotations exist for stabilizing control pulses against operational imperfections [14], they are hard to reconcile with the b.b. framework, which does not easily lend itself to incorporating robustness features.

In this Letter, we overcome the shortcomings of the b.b. formulation by showing how to implement dynamical decoupling using bounded-strength Hamiltonians. If $\mathcal{G}$ is the discrete group specifying the desired b.b. decoupler, the key idea is to constrain the motion of the control propagator during each cycle along a path that interpolates between the elements of $\mathcal{G}$. Under mild assumptions on the control Hamiltonians, a decoupling prescription inducing the same symmetry structure as in the b.b. limit can be constructed by exploiting *Eulerian cycles on a Cayley graph of* $\mathcal{G}$. In addition to significantly weakening the relevant implementation requirements, Eulerian decoupling turns out to be largely insensitive to control faults, opening the way to the robust dynamical generation of noise-protected subsystems.

Decoupling setting.—Let the target system S be defined on a finite-dimensional state space $\mathcal{H}_S$, and let $\mathrm{End}(\mathcal{H}_S)$ be the corresponding operator algebra. Thus, $\mathcal{H}_S \simeq \mathbb{C}^d$, $\mathrm{End}(\mathcal{H}_S) \simeq \mathrm{Mat}_d(\mathbb{C})$ for some d, with $d = 2^n$ for an n-qubit system. S may be coupled to an uncontrollable environment E, whereby the evolution on the joint state space $\mathcal{H}_S \otimes \mathcal{H}_E$ is ruled by a total drift Hamiltonian $H_0 = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E + \sum_\alpha S_\alpha \otimes E_\alpha$ for traceless noise generators $S_\alpha \in \mathrm{End}(\mathcal{H}_S)$ [3]. A decoupling problem is concerned with characterizing the effective evolutions that can be generated from H_0 via the application of a control field $H_c(t) \otimes \mathbb{1}_E$ acting on S alone [3]. Let $U_c(t) = \mathcal{T}\exp\{-i \int_0^t dt' H_c(t')\}$ denote the control propagator, with $\hbar = 1$. In a frame that removes the control field, the dynamics is governed by a time-dependent Hamiltonian $\tilde{H}(t) = U_c^\dagger(t) H_0 U_c(t)$, and the overall evolution in the Schrödinger picture results from the net propagator

$$U(t) = U_c(t)\mathcal{T}\exp\left\{-i \int_0^t dt' \tilde{H}(t')\right\}. \quad (1)$$

Assuming that the control action is cyclic, $U_c(t + T_c) = U_c(t)$ for some cycle time $T_c > 0$ and for all t, the stroboscopic dynamics $U(t_M = MT_c)$, $M \in \mathbb{N}$, can be identified with the effective evolution induced by $\tilde{H}(t)$ in (1). First-order decoupling aims at generating the desired evolution to lowest order in T_c, $U(t_M) = \exp(-i\overline{H}^{(0)} t_M)$, where

$$\overline{H}^{(0)} = \frac{1}{T_c} \int_0^{T_c} dt' U_c^\dagger(t') H_0 U_c(t'). \quad (2)$$

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While higher-order corrections can be systematically evaluated, the approximation (2) tends to become exact as the fast control limit $T_c \to 0$ is approached [1,3,5].

In the simplest b.b. decoupling setting, the time average in (2) maps directly into a group-theoretical average. Let $\mathcal{G}$ be a discrete group of order $|\mathcal{G}| > 1$, $\mathcal{G} = \{g_j\}$, $j = 0, \ldots, |\mathcal{G}| - 1$, acting on $\mathcal{H}_S$ via a faithful, unitary, projective representation μ, $\mu(\mathcal{G}) \subset \mathcal{U}(\mathcal{H}_S)$. Let images of abstract quantities under μ be denoted as $\mu(g_j) = \hat{g}_j$, and so forth [15]. Then b.b. decoupling according to $\mathcal{G}$ is implemented by specifying $U_c(t)$ over each of the $|\mathcal{G}|$ equal subintervals defining a control cycle [3]:

$$U_c[(\ell - 1)\Delta t + s] = \hat{g}_{\ell-1}, \qquad s \in [0, \Delta t), \tag{3}$$

with $T_c = |\mathcal{G}|\Delta t$ for $\Delta t > 0$, and $\ell = 1, \ldots, |\mathcal{G}|$. The resulting control action corresponds to extracting the $\mathcal{G}$-invariant component of H_0, $\overline{H}^{(0)} = \Pi_{\hat{\mathcal{G}}}(H_0)$, where

$$\Pi_{\hat{\mathcal{G}}}(X) = \frac{1}{|\mathcal{G}|} \sum_{g_j \in \mathcal{G}} \hat{g}_j^\dagger X \hat{g}_j, \qquad X \in \mathrm{End}(\mathcal{H}_S), \tag{4}$$

is the projector onto the commutant $\widehat{\mathbb{C}\mathcal{G}}'$ of $\widehat{\mathbb{C}\mathcal{G}}$ in $\mathrm{End}(\mathcal{H}_S)$ [3,4]. Thus, $U_c(t)$ jumps from $\hat{g}_{\ell-1}$ to $\hat{g}_\ell = (\hat{g}_\ell \hat{g}_{\ell-1}^\dagger)\hat{g}_{\ell-1}$ through the application of an arbitrarily strong, instantaneous control kick at the ℓth end point $t_\ell = \ell\Delta t$, realizing the b.b. pulse $\hat{p}_\ell = \hat{g}_\ell \hat{g}_{\ell-1}^\dagger$ [5].

Eulerian dynamical decoupling.—We seek a way for smoothly steering $U_c(t)$ from $\hat{g}_{\ell-1}$ to $\hat{g}_\ell$ by a control action distributed along the whole ℓth subinterval. Let $\Gamma = \{\gamma_\lambda\}$, $\lambda = 1, \ldots, |\Gamma|$, be a generating set for $\mathcal{G}$. The *Cayley graph* $G(\mathcal{G}, \Gamma)$ of $\mathcal{G}$ with respect to Γ is the directed multigraph whose edges are colored with the generators [16], where vertex $g_{\ell-1}$ is joined to vertex g_ℓ by an edge of color λ if and only if $g_\ell g_{\ell-1}^{-1} = \gamma_\lambda$, i.e., $g_\ell = \gamma_\lambda g_{\ell-1}$. Physically, imagine that we have the ability to implement each generator $\hat{\gamma}_\lambda$, by the application of control Hamiltonians $h_\lambda(t)$ over Δt,

$$\hat{\gamma}_\lambda = \mathcal{T}\exp\left\{-i \int_0^{\Delta t} dt' h_\lambda(t')\right\}, \qquad \lambda = 1, \ldots, |\Gamma|. \tag{5}$$

The choice of $h_\lambda(t)$ is not unique, allowing for implementation flexibility. Once a choice is made, the control action is determined by assigning a cycle time and a rule for switching the Hamiltonians $h_\lambda(t)$ during the cycle subintervals. We show how a useful rule results from sequentially implementing generators so that they follow a *Eulerian cycle* on $G(\mathcal{G}, \Gamma)$. A Eulerian cycle is defined as a cycle that uses each edge exactly once [16]. Because a Cayley graph is regular, it always possesses Eulerian cycles, having length $L = |\mathcal{G}||\Gamma|$ [16].

Let a Eulerian cycle beginning at the identity g_0 of G be given by the sequence of edge colors used, $\mathcal{P}_E = (p_1, p_2, \ldots, p_L)$, with $p_\ell = \gamma_\lambda$ for some λ, for every ℓ. Note that each vertex has exactly one departing edge of each color, so that $\mathcal{P}_E$ determines a well-defined path. We define Eulerian decoupling according to $\mathcal{G}$ by letting $T_c = L\Delta t$ and by assigning $U_c(t)$ as follows:

$$U_c[(\ell - 1)\Delta t + s] = u_\ell(s) U_c[(\ell - 1)\Delta t], \tag{6}$$

where $s \in [0, \Delta t)$, and $u_\ell(s) = \mathcal{T}\exp\{-i \int_0^s dt' h_\ell(t')\}$, $u_\ell(\Delta t) = \hat{p}_\ell$, $\ell = 1, \ldots, L$. This decoupling prescription means that during the ℓth subinterval one chooses as a control Hamiltonian the one that implements the generator $\hat{\gamma}_\lambda$, with γ_λ coloring the edge p_ℓ in $\mathcal{P}_E$. The effective Hamiltonian $\overline{H}^{(0)}$ is computed by inserting (6) in (2). The L terms of the time average can be partitioned into $|\Gamma|$ families, each corresponding to a fixed generator γ_λ. Because $\mathcal{P}_E$ contains exactly one γ_λ-colored edge ending at any given vertex g_j, each family effects a sum over the group elements as in (4). The final result can be expressed as $\overline{H}^{(0)} = \mathcal{Q}_{\hat{\mathcal{G}}}(H_0)$, where

$$\mathcal{Q}_{\hat{\mathcal{G}}}(X) = \Pi_{\hat{\mathcal{G}}}[F_{\hat{\Gamma}}(X)], \qquad X \in \mathrm{End}(\mathcal{H}_S), \tag{7}$$

and the map $F_{\hat{\Gamma}}$ implements an average over both the group generators and the control subinterval:

$$F_{\hat{\Gamma}}(X) = \frac{1}{|\Gamma|} \sum_{\lambda=1}^{|\Gamma|} \frac{1}{\Delta t} \int_0^{\Delta t} ds\, u_\lambda^\dagger(s) X u_\lambda(s). \tag{8}$$

Thanks to the way $\Pi_{\hat{\mathcal{G}}}$ enters (7), $\mathcal{Q}_{\hat{\mathcal{G}}}(X) \in \widehat{\mathbb{C}\mathcal{G}}'$ for an arbitrary input X. This property will be repeatedly used in the following. The link between Eulerian decoupling and $\mathcal{G}$ symmetrization is established upon enforcing some additional compatibility between $\Pi_{\hat{\mathcal{G}}}$ and $F_{\hat{\Gamma}}$.

Theorem.—Let X be any (time-independent) operator on $\mathcal{H}_S$, and let $\mathcal{Q}_{\hat{\mathcal{G}}}$ be defined as above. If the controls are chosen in the decoupling group algebra, $h_\ell(t) \in \widehat{\mathbb{C}\mathcal{G}}$ for all $t \in [0, \Delta t]$ and for all $\ell = 1, \ldots, L$, then

$$\mathcal{Q}_{\hat{\mathcal{G}}}(X) = \Pi_{\hat{\mathcal{G}}}(X), \qquad X \in \mathrm{End}(\mathcal{H}_S).$$

Proof.—The assumption on the controls implies that $u_\lambda(s) \in \widehat{\mathbb{C}\mathcal{G}}$ $\forall \lambda$, $\forall s \in [0, \Delta t]$. Thus, $F_{\hat{\Gamma}}(Y) = Y$ for every (time-independent) $Y \in \widehat{\mathbb{C}\mathcal{G}}'$. Now let $X \in \mathrm{End}(\mathcal{H}_S)$ and calculate $\mathcal{Q}^2_{\hat{\mathcal{G}}}(X) = \Pi_{\hat{\mathcal{G}}}[\mathcal{Q}_{\hat{\mathcal{G}}}(X)] = \Pi_{\hat{\mathcal{G}}}[F_{\hat{\Gamma}}(X)] = \mathcal{Q}_{\hat{\mathcal{G}}}(X)$. Thus, $\mathcal{Q}_{\hat{\mathcal{G}}}$ is a projector. Because $\mathrm{Range}\,\mathcal{Q}_{\hat{\mathcal{G}}} \subseteq \widehat{\mathbb{C}\mathcal{G}}'$, $\mathcal{Q}_{\hat{\mathcal{G}}} = \Pi_{\hat{\mathcal{G}}}$ iff $\mathcal{Q}_{\hat{\mathcal{G}}}$ has identity action on $\widehat{\mathbb{C}\mathcal{G}}'$. Let $Y \in \widehat{\mathbb{C}\mathcal{G}}'$; then $\mathcal{Q}_{\hat{\mathcal{G}}}(Y) = \Pi_{\hat{\mathcal{G}}}[F_{\hat{\Gamma}}(Y)] = \Pi_{\hat{\mathcal{G}}}(Y)$. ■

The b.b. limit is formally recovered by letting $F_{\hat{\Gamma}}$ be the identity map. In the Eulerian approach, at the expense of lengthening the control cycle by a factor of $|\Gamma|$, the same $\mathcal{G}$ symmetrization can be attained using *bounded* controls. The maximum strengths achievable in implementing the generators (5) directly affect the minimum attainable T_c and, therefore, the accuracy of the averaging [3]. While the overhead $|\Gamma|$ depends on the specific group, it is worth noting that, similar to $\Pi_{\hat{\mathcal{G}}}$ [4], $\mathcal{Q}_{\hat{\mathcal{G}}}$ satisfies the property that $\mathcal{Q}_{\hat{\mathcal{G}}}(X) = \mathcal{Q}_{\widehat{\mathcal{G}/\mathcal{G}_0}}(X)$ whenever $\mathcal{G}_0$ is a normal subgroup of $\mathcal{G}$ and $X \in \widehat{\mathbb{C}\mathcal{G}_0}'$ [17]. Thus, if the dynamics is already $\mathcal{G}_0$ invariant, Eulerian decoupling according to $\mathcal{G}$ can be accomplished by using a Cayley graph of the smaller quotient group $\mathcal{G}/\mathcal{G}_0$.

Robustness analysis.—The fact that the control actions are distributed along finite time intervals translates into major gains in terms of resilience of Eulerian schemes against operational imperfections. Imagine that systematic implementation errors result in a faulty control Hamiltonian $H_c'(t)$, and partition $H_c'(t)$ into

$$H_c'(t) = H_c(t) + \Delta H_c(t), \tag{9}$$

such that $H_c(t) \in \widehat{\mathbb{C}\mathcal{G}}$ is the intended control Hamiltonian, and $\Delta H_c(t)$ is the error component. Now work in the same frame used earlier, which removes only the ideal control part from the effective Hamiltonian. Because $H_0 + H_c'(t) = [H_0 + \Delta H_c(t)] + H_c(t)$, this maps the evolution under H_0 with the faulty control $H_c'(t)$ into the evolution under $H_0 + \Delta H_c(t)$ with the ideal control. Thus, the new effective dynamics may be obtained by replacing H_0 with $H_0 + \Delta H_c(t)$ in (2).

Suppose that the faults are properly correlated with the underlying path, meaning that every time a given generator $\hat{\gamma}_\lambda$ is implemented, the same imperfection occurs at equivalent temporal locations within the subinterval, regardless of the position of γ_λ along $\mathcal{P}_E$. Then $\Delta H_c[(\ell - 1)\Delta t + s] = \Delta h_\lambda(s)$, λ being the color of the edge that $\mathcal{P}_E$ uses during the ℓth subinterval. By a calculation similar to the ideal case, $\mathcal{Q}_{\hat{\mathcal{G}}}$ is modified as follows:

$$\mathcal{Q}'_{\hat{\mathcal{G}}}(X) = \Pi_{\hat{\mathcal{G}}}(X) + \mathcal{Q}_{\hat{\mathcal{G}}}(\Delta H_c), \tag{10}$$

where $\mathcal{Q}_{\hat{\mathcal{G}}}(\Delta H_c)$ can be computed as in (7) and (8), but with the operator X in the integral replaced by one that depends on s and λ. Thus, $\mathcal{Q}_{\hat{\mathcal{G}}}(\Delta H_c)$ is a functional of the fault history over $[0, \Delta t]$ and characterizes the residual control errors experienced by the system. Notably, two useful features emerge: without extra assumptions, any residual control error belongs to $\widehat{\mathbb{C}\mathcal{G}}'$. If, in addition, $\Delta H_c(t)$ is itself [as $H_c(t)$] in $\widehat{\mathbb{C}\mathcal{G}}$, then all control effects remain in $\widehat{\mathbb{C}\mathcal{G}}$, and the residual control errors belong to the so-called center $Z_{\widehat{\mathbb{C}\mathcal{G}}} = \widehat{\mathbb{C}\mathcal{G}} \cap \widehat{\mathbb{C}\mathcal{G}}'$.

The effects of $\mathcal{Q}_{\hat{\mathcal{G}}}(\Delta H_c)$ may still adversely impact the system. However, they can be compensated for by encodings in appropriate subsystems [6]. Let $J \in \mathcal{J}$ label the irreducible components of $\widehat{\mathbb{C}\mathcal{G}}$. Then $\mathcal{H}_S$ can be represented as

$$\mathcal{H}_S \simeq \oplus_J \mathcal{H}_J \simeq \oplus_J \mathcal{C}_J \otimes \mathcal{D}_J \simeq \oplus_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}, \tag{11}$$

with $n_J, d_J \in \mathbb{N}$, $\sum_J n_J d_J = d$, and the action of the decoupling group algebra and its commutant given by $\widehat{\mathbb{C}\mathcal{G}} \simeq \oplus_J \mathbb{1}_{n_J} \otimes \mathrm{Mat}_{d_J}(\mathbb{C})$, $\widehat{\mathbb{C}\mathcal{G}}' \simeq \oplus_J \mathrm{Mat}_{n_J}(\mathbb{C}) \otimes \mathbb{1}_{d_J}$, respectively. Because both $\Pi_{\hat{\mathcal{G}}}(S_\alpha)$ and $\mathcal{Q}_{\hat{\mathcal{G}}}(\Delta H_c)$ are in $\widehat{\mathbb{C}\mathcal{G}}'$, $\mathcal{D}_J$ subsystems are noiseless and their dynamical generation robust regardless of whether $\Delta H_c(t)$ belongs to $\widehat{\mathbb{C}\mathcal{G}}$ or not. This applies, in particular, if $\mathcal{G}$ acts irreducibly on $\mathcal{H}_S$, in which case a robust implementation of maximal decoupling is achievable by averaging over a nice error basis on $\mathbb{C}^d$ [3]. In fact, encoding into $\mathcal{D}_J$ subsystems may be valuable even in situations where the assumption that the controls are in $\widehat{\mathbb{C}\mathcal{G}}$ cannot be met: as $\mathcal{Q}_{\hat{\mathcal{G}}}(S_\alpha) \in \widehat{\mathbb{C}\mathcal{G}}'$, $\mathcal{D}_J$ subsystems remain unaffected by the noise. Note that for such subsystems, both the implementation of the decoupling scheme and the execution of encoded control operations are to be effected through fast modulation of Hamiltonians along the control cycle [5,6].

Whenever $\mathcal{Q}_{\hat{\mathcal{G}}}(\Delta H_c)$ originates from faults in $\widehat{\mathbb{C}\mathcal{G}}$, additional options are viable. If the representation μ is primary, i.e., $Z_{\widehat{\mathbb{C}\mathcal{G}}} = \mathbb{C}\mathbb{1}$, then *any* systematic error is effectively eliminated, and no encoding is necessary as long as noise suppression is ensured, that is, $\Pi_{\hat{\mathcal{G}}}(S_\alpha) = 0$ for all α. If μ is not primary, then elements in the center are diagonal over each irreducible component. Thus, encodings into either a $\mathcal{H}_J$ subspace or a $\mathcal{C}_J$ subsystem are insensitive to the control faults and protected against the noise generators if $\Pi_{\hat{\mathcal{G}}}(S_\alpha) \in Z_{\widehat{\mathbb{C}\mathcal{G}}}$ as well. In practice, choosing a $\mathcal{C}_J$ subsystem may be especially appealing, because not only is universal encoded control achievable by less-demanding, slow application of Hamiltonians in $\widehat{\mathbb{C}\mathcal{G}}'$ [5], but added robustness against *arbitrary* control errors in $\widehat{\mathbb{C}\mathcal{G}}$ is automatically provided [6]. Next, we outline some applications relevant to QIP.

Example 1: Eulerian Carr-Purcell decoupling on a qubit.—Consider a single decohering qubit, $\{S_\alpha\} = \{\sigma_z\}$ [2]. The decoupling group $\mathcal{G} = \mathcal{Z}_2 = \{0, 1\}$ is represented in $\mathcal{U}(\mathbb{C}^2)$ as $\hat{\mathcal{G}} = \{\mathbb{1}, \sigma_x\}$. There is one generator, $\gamma_1 = 1$; hence $L = 2$ with no overhead with respect to the b.b. case. Let $u_x(s) = \mathcal{T}\exp\{-i \int_0^s dt' h_x(t')\}$ for any Hamiltonian $h_x(t) \in \widehat{\mathbb{C}\mathcal{G}}$ realizing $\hat{\gamma}_1 = \sigma_x = u_x(\Delta t)$. On $G(\mathcal{G}, \Gamma)$ choose $\mathcal{P}_E = (\gamma_1, \gamma_1)$. Then Eulerian decoupling is accomplished by letting $U_c(t) = u_x(t)$, for $t \in [0, \Delta t)$, and $U_c(t) = u_x(s)\sigma_x$ for $t \in [\Delta t, \Delta t + s)$, $s \in [0, \Delta t)$. By explicit calculation of $\mathcal{Q}_{\hat{\mathcal{G}}}(\Delta H_c)$, one sees that systematic errors along σ_y, σ_z produce no effect. Elimination of residual control errors in $Z_{\widehat{\mathbb{C}\mathcal{G}}}$ requires using the full Pauli group.

Example 2: Eulerian Pauli decoupling on qubits.—Let $\hat{\mathcal{G}} = \{\mathbb{1}, X, Y, Z\}$ be the Pauli error basis for a qubit, with $X = \sigma_x$, $Z = \sigma_z$, and $Y = XZ$. This corresponds to $\mathcal{G} = \mathcal{Z}_2 \times \mathcal{Z}_2$, projectively represented in $\mathcal{U}(\mathbb{C}^2)$. $\mathcal{G}$ has two generators, e.g., $\gamma_1 = (0, 1)$, $\gamma_2 = (1, 0)$, realized as $\hat{\gamma}_1 = X$, $\hat{\gamma}_2 = Z$, respectively. A Eulerian path on $G(\mathcal{G}, \Gamma)$ is $\mathcal{P}_E = (\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_2, \gamma_1, \gamma_2, \gamma_1)$, of length $L = 8$. The assumption that both h_λ and Δh_λ, $\lambda = 1, 2$, are in $\widehat{\mathbb{C}\mathcal{G}}$ is automatically satisfied, as $\widehat{\mathbb{C}\mathcal{G}} = \mathrm{Mat}_2(\mathbb{C})$. Then (6) results into a robust implementation of maximal averaging, $\Pi_{\hat{\mathcal{G}}}(\sigma_u) = 0$, $u = x, y, z$. For n qubits,

$\mathcal{G} = Z_d \times Z_d$, with $d = 2^n$. Thus, $|\mathcal{G}| = 4^n$ and since two generators are needed for each qubit, $L = n2^{2n+1}$, causing the procedure to be (as in the b.b. limit [3]) inefficient.

Example 3: Eulerian collective spin-flip decoupling.—For n qubits, let $\mathcal{G} = Z_2 \times Z_2$ act via the n-fold tensor power representation in $\mathcal{U}[(\mathbb{C}^2)^{\otimes n}]$, which is projective for n odd and regular for n even. For any n, $\hat{\mathcal{G}} = \{\mathbb{1}, X, Y, Z\}$, where $X = \otimes_{k=1}^n \sigma_x^{(k)}$, $Z = \otimes_{k=1}^n \sigma_z^{(k)}$, and $Y = XZ$. Decoupling according to $\mathcal{G}$ averages out arbitrary linear noise, $\Pi_{\hat{\mathcal{G}}}(S_\alpha) = 0$, $S_\alpha \in \text{span}\{\sigma_u^{(k)}\}$ [5]. For Eulerian implementation, the same path of Example 2 may be used, under the appropriate realization of the collective generators $\hat{\gamma}_1 = X$, $\hat{\gamma}_2 = Z$. Ensuring $\mathcal{G}$ symmetrization requires that the control Hamiltonians $h_{1,2}(t) \in \widehat{\mathbb{C}\mathcal{G}}$. Because both $\widehat{\mathbb{C}\mathcal{G}}'$ and $Z_{\widehat{\mathbb{C}\mathcal{G}}}$ are nontrivial, residual control errors may arise due to $\mathcal{Q}_{\hat{\mathcal{G}}}(\Delta H_c)$. The situation is simpler for n even, as $\widehat{\mathbb{C}\mathcal{G}}$ is Abelian hence supporting four $(n-2)$-dimensional irreducible subspaces $\mathcal{H}_J$. Besides being noiseless in the decoupling limit and insensitive to arbitrary control errors in $\widehat{\mathbb{C}\mathcal{G}}$, encoding into a $\mathcal{H}_J$ subspace is further motivated by the possibility to achieve encoded universality via slow application of two-body Hamiltonians in $\widehat{\mathbb{C}\mathcal{G}}'$ [6]. For n odd, both $\mathcal{C}_J$ and $\mathcal{D}_J$ factors may occur. Leaving aside details here, we note that $\mathcal{D}_J$ subsystems may be useful if implementing $\hat{\gamma}_1, \hat{\gamma}_2$ via Hamiltonians in $\widehat{\mathbb{C}\mathcal{G}}$ is difficult in practice.

Example 4: Eulerian symmetric decoupling.—Let $\mathcal{G} = S_n$ be the symmetric group of order n, acting on $\mathcal{H}_S \simeq (\mathbb{C}^2)^{\otimes n}$ via $\hat{g}_j \otimes_{k=1}^n |\psi_k\rangle = \otimes_{k=1}^n |\psi_{g_j(k)}\rangle$, $g_j \in S_n$. In particular, the action implementing a transposition $(k-1\,k)$, $k \in \{1, \ldots, n\}$, effects an exchange gate between qubits $k-1, k$, denoted by $\text{SWAP}_{k-1,k}$. Symmetric decoupling enables one, in principle, to engineer collective error models on S starting from arbitrary linear interactions between S and E [4,6]. A minimal generating set for S_n is given by $\gamma_1 = (1,2)$, $\gamma_2 = (1,2,\ldots,n)$, i.e., an adjacent transposition and the cyclic shift, respectively. $\widehat{\mathbb{C}S_n}$ contains the Heisenberg couplings $h(k,l) = \vec{\sigma}_k \cdot \vec{\sigma}_l$. In fact, every operator in $\widehat{\mathbb{C}S_n}$ can be realized by applying Heisenberg Hamiltonians [18]. Focus, for instance, on S_3 symmetrization, which may be relevant for inducing collective decoherence on blocks of three qubits [19]. Then $\hat{\gamma}_1 = \text{SWAP}_{1,2}$ and $\hat{\gamma}_2 = \text{SWAP}_{1,2}\text{SWAP}_{2,3}$, with $L = 12$. Because $\exp[-i\pi h(k,l)/4] = \text{SWAP}_{k,l}$, $\hat{\gamma}_1$ can be implemented by choosing $h_1 = a_1 h(1,2)$, with strength $a_1 = \pi/4\Delta t$, while $\hat{\gamma}_2$ can be realized by a piecewise-constant Hamiltonian $h_2(t) = a_2 h(2,3)$ for $t \in [0, \Delta t/2)$, $h_2(t) = a_2 h(1,2)$ for $t \in [\Delta t/2, \Delta t]$, $a_2 = \pi/2\Delta t$. A Eulerian path on $G(S_3, \Gamma)$ is $\mathcal{P}_E = (\gamma_2, \gamma_2, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_1, \gamma_2, \gamma_1, \gamma_1, \gamma_2, \gamma_1)$. Eulerian decoupling (6) then allows for a robust dynamical generation of the smallest nontrivial noiseless subsystem [20,21], supported by a factor $\mathcal{D}_J \simeq \mathbb{C}^2$ carrying the two-dimensional irreducible component $J = [2\,1]$ of S_3.

Conclusion.—We developed an approach to dynamical decoupling that combines the group-theoretical essence of the b.b. setting with graph-theoretical control design according to Eulerian cycles. Besides allowing for considerable leeway in the physical implementation of the basic control generators, Eulerian decoupling eliminates the need for unfeasible b.b. pulses and naturally incorporates robustness against realistic control faults. While relaxing the timing constraints of the required controls clearly emerges as the main remaining challenge in decoupling design, we believe that our results constitute a significant advance toward constructing practical decoupling schemes and may motivate novel approaches to robust control and simulation of quantum systems.

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[17] Here, the set of operators $\widehat{\mathcal{G}/\mathcal{G}_0}$ is defined with respect to an arbitrary choice of coset representatives of $\mathcal{G}_0$ and is not necessarily a projective representation.
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