

## Temperature-Transformed “Minimal Coupling”: Magnetofluid Unification

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The dynamics of a relativistic, hot charged fluid is expressed in terms of a hybrid magnetofluid field which unifies the electromagnetic field with an appropriately defined but analogous flow field. The unification is affected by a well-defined prescription that allows the derivation of the equations of motion of a plasma embedded in an electromagnetic field from the field-free equations. The relationship of this prescription with the minimal coupling prescription of particle dynamics is discussed; the changes brought about by the plasma temperature are highlighted. A few consequences of the unification are worked out.

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The minimal coupling prescription, epitomized in the substitution,

$$p_\mu \rightarrow p_\mu + qA_\mu, \quad mU_\mu \rightarrow mU_\mu + qA_\mu \quad (1)$$

reproduces the Lorentz force correctly, and is therefore routinely used for incorporating the electromagnetic (EM) field in particle dynamics. In Eq. (1),  $p_\mu(U_\mu)$  and  $A_\mu$ , are, respectively, the particle four momentum (four-velocity) and the electromagnetic four potential,  $q$  is the electric charge and  $m$  is the mass of the particle.

In this Letter I examine two related questions:

(i) Does there exist an equivalent recipe (like the minimal coupling prescription) for deriving the equations of motion of a hot charged fluid immersed in an EM field from the equations it obeys in the absence of the EM field? What is the detailed nature of this prescription and how does it reflect the effect of statistical attributes like the temperature which are absent in particle dynamics?

(ii) Is it possible to cast the fluid velocity field in clothing designed to fit the electromagnetic field and does such a casting lead to a “unification” of the flow field with electromagnetism? And will the new structure be mathematically simpler and more revealing than the standard formalism?

For this enquiry, I deal with a standard two-species plasma (one positively and one negatively charged), although generalization to many species is straightforward. A local Maxwellian closure will ensure that in the ensuing description, the scalar pressure, along with the four flux (constructed from density and three components of the velocity) constitutes the entire set of fluid variables. In addition to Maxwell’s equations

$$\partial_\nu F^{\mu\nu} = J^\mu, \quad (2)$$

$$\partial_\nu \mathcal{F}^{\mu\nu} = 0, \quad (3)$$

the system consists of the following fluid equations valid for each species ( $\alpha$  is the species index): the continuity equation

$$\partial_\nu \Gamma_{(\alpha)}^\nu = 0, \quad (4)$$

and the equation of motion

$$\partial_\nu T_{(\alpha)}^{\mu\nu} = q_\alpha F^{\mu\nu} \Gamma_{\nu(\alpha)}, \quad (5)$$

where  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$  is the fully antisymmetric electromagnetic field tensor related to the electric and magnetic fields through

$$\begin{aligned} F^{0i} &= E^i \\ F^{ij} &= B^k, \\ i, j, k &\text{ cyclic,} \end{aligned} \quad (6)$$

and  $\mathcal{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$  is its dual obtained by the substitution  $\mathcal{F}^{\mu\nu} = F^{\mu\nu}(\mathbf{E} \rightarrow \mathbf{B}, \mathbf{B} \rightarrow -\mathbf{E})$ . The fully relativistic energy-momentum tensor is given by [1,2]

$$T_{(\alpha)}^{\mu\nu} = p\eta^{\mu\nu} + h_\alpha U_{(\alpha)}^\mu U_{(\alpha)}^\nu \quad (7)$$

where  $p_\alpha$  is the pressure,  $h_\alpha$  is the enthalpy density, and  $U_{(\alpha)}^\mu = \{\gamma_\alpha, \gamma_\alpha \mathbf{V}\} \equiv \{\gamma_\alpha, \mathbf{U}_\alpha\}$  is the four-velocity of the  $\alpha$ th component of the fluid and  $\gamma = (1 - V^2)^{-1/2}$  is the relativistic factor. The flux

$$\Gamma_{(\alpha)}^\mu = n_{R(\alpha)} U_{(\alpha)}^\mu, \quad (8)$$

where  $n_R$  is a scalar measuring the density in the rest-frame (local) of the given fluid. The current

$$J^\mu = \sum_\alpha q_\alpha \Gamma_{(\alpha)}^\mu = \sum_\alpha q_\alpha n_{R\alpha} U_{(\alpha)}^\mu \quad (9)$$

is the input needed to couple the fluid to Maxwell’s equations. Notice that summing over species converts (5) into  $[\mathcal{T}^{\mu\nu} = \sum_\alpha T_{(\alpha)}^{\mu\nu}]$

$$\partial_\nu \mathcal{T}^{\mu\nu} = F^{\mu\nu} J_\nu \quad (10)$$

which, in principle, can be solved to find  $J_\nu$ . From now on, I will drop the species index, and much of what follows will hold for each species.

As stated earlier, I will use the standard, local Maxwellian closure, which, for the relativistic plasmas,

yields the enthalpy density [2],

$$h = mn_R \frac{K_3(m/T)}{K_2(m/T)} \equiv mn_R f(T), \quad (11)$$

where  $m$  is the rest mass of the particles constituting the  $\alpha$ th component of the fluid and  $K_n$ 's are modified Bessel functions of the second kind with the argument  $\zeta = m/T$ . The function  $f(T) \equiv f$  is a function only of the temperature. The theory being developed is fully relativistic, both in temperature (arbitrary  $T/mc^2 \equiv T/m$ ) and in the directed speed.

After these necessary preliminaries, I now introduce, in complete analogy to the EM field tensor, the fully antisymmetric second-rank “flow” tensor

$$S^{\mu\nu} = \partial^\mu f U^\nu - \partial^\nu f U^\mu \quad (12)$$

constructed from the kinematic ( $U_\mu$ ) and the statistical [ $f(T)$ ] attributes of the fluid. Its nonzero components [Greek (Latin) indices go from 0–3(1–3)] are

$$S^{0i} = Q^i, \quad (13)$$

$$S^{ij} = R^k, \quad i, j, k \text{ cyclic.} \quad (14)$$

The three-vectors  $\mathbf{Q}$  and  $\mathbf{R}$  are the fluid equivalents of the electric and the magnetic fields [ $\mathbf{U} = \gamma\mathbf{V}$ ]

$$\mathbf{Q} = -\left[\frac{\partial}{\partial t} f \mathbf{U} + \nabla f \gamma\right], \quad (15)$$

$$\mathbf{R} = \nabla \times f \mathbf{U}. \quad (16)$$

The reason for introducing the weight factor  $f$  in the definitions will become clear later. By construction  $\mathbf{Q}$  and  $\mathbf{R}$  satisfy the equivalent of the homogeneous Maxwell's equations [Eq. (3)]  $\Rightarrow \nabla \cdot \mathbf{B} = 0, \partial \mathbf{B} / \partial t + \nabla \times \mathbf{E} = 0$ ,

$$\nabla \cdot \mathbf{R} = 0 \quad (17)$$

$$\frac{\partial \mathbf{R}}{\partial t} + \nabla \times \mathbf{Q} = 0, \quad (18)$$

which could be expressed explicitly covariantly

$$\partial_\nu \Sigma^{\mu\nu} = 0, \quad (19)$$

where  $\Sigma^{\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta} S_{\alpha\beta}$  is the dual of  $S^{\mu\nu}$ . Until now I have just given a set of self-consistent definitions. Now I will work on the equation of motion (5) to express it in terms of  $S^{\mu\nu}$ .

Using Eq. (11) for  $h$ , we could spell out Eq. (5) as

$$\partial^\nu p + mn_R U^\mu \partial_\mu f U^\nu = qn_R F^{\nu\mu} U_\mu, \quad (20)$$

where we have invoked  $\partial_\mu \Gamma^\mu = \partial_\mu n_R U^\mu = 0$ . Contracting (12) with  $U_\mu$ , we obtain

$$U^\mu \partial_\mu f U^\nu = U_\mu S^{\mu\nu} - \partial^\nu f. \quad (21)$$

Substituting (21) into (20) and combining terms contain-

ing  $p$  and  $f$ ,

$$T \partial^\nu \sigma = q U_\mu M^{\nu\mu} \quad (22)$$

with

$$M^{\nu\mu} = F^{\nu\mu} + (m/q) S^{\nu\mu} \quad (23)$$

representing the effective field tensor combining the electromagnetic and fluid “forces.” The entropy-like quantity  $\sigma = \ell n[(p/K_2)(m/T)^2 \exp[-(m/T)(K_3/K_2)]]$  is an expression of purely fluid (thermal) attributes. From the indicial asymmetry of  $M$  (22), we find that  $T U_\nu \partial^\nu \sigma = \gamma T d\sigma/dt = 0$  ( $d/dt = \partial/\partial t + \mathbf{V} \cdot \nabla$ ) is the total time derivative leading to the well-known isentropic equation of state  $\sigma = \text{const}$  [2]. But this constancy is along the fluid streamlines and in general the entropy  $\sigma$  is space-time dependent.

Let us first examine the fate of a fluid for which  $\sigma$  is a global constant (homotropic fluid). This special case of the homotropic fluid is of great importance in the theory of plasma self-organization; it is the most general system (others have either constant density or constant temperature) in which the pressure force in the nonrelativistic equation of motion becomes a full gradient. This, in turn, allows the equation to be cast in a vortex dynamics form essential for the existence of the constant of motion which lie at the foundation of self-organization [3,4]. For the homotropic fluid, then, the equation of motion becomes

$$U_\mu M^{\nu\mu} = 0, \quad (24)$$

a remarkably simple and revealing form. Remembering that  $U_\mu F^{\nu\mu}$  is the expression for the electromagnetic force, (24) tells us that the “magnetofluid” field  $M^{\nu\mu}$ , obtained from the unification of the flow field and the electromagnetic field, exerts no net force on the fluid; the entire complicated fluid dynamics of relativistic charged particles in an electromagnetic field is contained in this rather innocent statement.

The construction of  $M^{\mu\nu}$  is the centerpiece of this effort. It is through  $M^{\mu\nu}$  that the flow field and the electromagnetic field are put on the same footing. Needless to say, that it is an inherent and fundamental property of the equation of motion; the formalism simply reveals the unity.

At this stage it is possible to partially answer the questions that motivated this effort. Through the construction of the fully antisymmetric tensor  $S^{\mu\nu}$ , we were, indeed, successful in dressing the fluid velocity field in electromagnetic clothing. This transformation, then, paved the way for the tensor  $S^{\mu\nu}$  to combine with the electromagnetic field tensor  $F^{\mu\nu}$  to yield the final unified field tensor  $M^{\nu\mu}$ .

To answer the first question we notice that a simple corollary of (24) is that, in the absence of the EM field, the fluid obeys

$$U_\mu S^{\nu\mu} = 0, \quad (25)$$

implying that the recipe to convert the field-free Eq. (25) to the equation of motion pertinent to the fluid immersed in the EM field (24) is contained in the substitution  $S^{\mu\nu} \rightarrow S^{\mu\nu} + (q/m)F^{\mu\nu}$  that readily follows from  $fU^\nu \rightarrow fU^\nu + (q/m)A^\nu$ , which for  $f = 1$ , would have been the minimum coupling prescription. For low temperatures,  $f$  does tend to unity and our prescription does approximately reduce to minimum coupling. As long as there is finite temperature, however, the statistical nature of the system (temperature being a statistical notion) becomes manifest and a purely inertial notion like the minimum coupling has to be suitably modified. The effective momentum  $\mathbf{p} = mf\gamma\mathbf{V}$  could be interpreted either in terms of an effective mass ( $mf$ ) or an effective relativistic factor ( $\gamma f$ ); either of these can be very different from its original value for plasmas with relativistic temperatures.

Before working out some of the consequences of Eq. (24), let me go back to the more general case of the isentropic fluid. After straightforward algebra (including using  $\gamma d/dt = U_\mu \partial^\mu$ ), we could rewrite Eq. (22) as

$$U_\mu H^{\nu\mu} = 0, \quad (26)$$

where the general magnetofluid tensor  $H = M + N$  has an additional term  $N^{\nu\mu} = (m/q)\{[\partial^\nu(T\sigma U^\mu) - \partial^\mu(T\sigma U^\nu)] - \sigma[\partial^\nu(TU^\mu) - \partial^\mu(TU^\nu)]\}$  which disappears when  $\sigma$  is a global constant. In analogy with the analysis following (24), Eq. (26) implies that the passage from the field-free dynamics to the dynamics with the EM field, is affected through the prescription  $[S + N] \rightarrow [S + N] + (m/q)F = (m/q)H$ , where  $H$  is the relevant unified field for this general case. It is difficult, however, to establish the relationship of this prescription (at the field level) to the minimal coupling because the tensor  $N$ , though fully antisymmetric, is not derivable from a potential as  $F$  is from  $A$  (or  $S$  is from  $fU$ ). The unification at only the field level suggests that the isentropic fluid will exhibit behavior similar to its special case for those aspects of plasma dynamics that do not depend explicitly upon the potential and depend only on the fields ( $\mathbf{E}$  and  $\mathbf{B}$ ).

Thus we find that there always exists a definite prescription for deriving the equations of motion of a plasma embedded in an arbitrary EM field from their field-free versions. Depending upon the magnitude and nature of the statistical attributes of the plasma (entropy and temperature), this prescription could differ from the minimal coupling prescription of particle dynamics. What is remarkable is that for a very important case of a homeotropic plasma the two prescriptions are essentially similar.

We now go back to derive some consequences of the unified dynamics represented by (24). Just to make sure

that the compact form (24) [or Eq. (26)] has all the familiar physics, let us take its nonrelativistic limit. This is easily accomplished by letting  $\gamma \rightarrow 1$ ,  $f \rightarrow 1$  everywhere except in the term  $\nabla f \gamma$  in  $\mathbf{Q}$ , where the first order terms are needed (leading order terms vanish). Using the large ( $m/T$ ) expansion of  $f \simeq 1 + 5/2(T/m)$ , and  $\gamma \simeq 1 + V^2/2$ , we find

$$\nabla f \gamma \rightarrow \nabla \left[ \frac{V^2}{2} + \frac{5T}{2m} \right], \quad (27)$$

which leads to the NR limit

$$m \frac{\partial \mathbf{V}}{\partial t} + \nabla \left[ m \frac{V^2}{2} + \frac{5T}{2} \right] = q \{ \mathbf{E} + \mathbf{V} \times [\mathbf{B} + (m/q)\nabla \times \mathbf{V}] \}, \quad (28)$$

precisely the standard equation of motion for a homeotropic fluid with the equation of state  $p \propto n^{5/3}$ , the non-relativistic limit of  $\sigma = \text{const}$ . It is interesting that the gradient forces seem to have been completely subsumed in the definition of the flow tensor  $S^{\mu\nu}$ .

To give a glimpse of the conceptual as well as calculational potential of this approach which assigns coprimacy to the flow and the electromagnetic field I will present a few consequences.

I will first delineate a procedure for deriving the bilinear constants of motion of the system. It is well known that for the electromagnetic field in vacuum, the helicity

$$h = \int A \cdot B d^3x \quad (29)$$

is a constant of the motion; it follows from the general notion in the field theories that the total ‘‘charge’’  $\int K^0 d^3x$  associated with a conserved four-vector

$$\partial_\mu K^\mu = 0 \quad (30)$$

is a constant of the motion, i.e.,  $(d/dt) \int K^0 d^3x = 0$ . It is easy to identify that the four-vector leading to helicity conservation is  $A_\mu \mathcal{F}^{\mu\nu}$ . Our unification naturally tells us that the equivalent invariants for the hot fluid will be obtained by replacing  $\mathcal{F}^{\mu\nu} \rightarrow \mathcal{M}^{\mu\nu}$  (the dual of  $M$ ) and  $A_\mu$  by  $A_\mu + (m/q)fU_\mu$ , the minimal coupling peculiar to hot fluids. Thus the vector

$$K^\mu = \left( A_\mu + \frac{m}{q} f U_\mu \right) \mathcal{M}^{\mu\nu} \quad (31)$$

will yield the constant of motion

$$G = \int d^3x K^0, \quad (32)$$

where

$$K^0 = [\mathbf{A} + (m/q)f\mathbf{U}] \cdot [\mathbf{B} + (m/q)\nabla \times f\mathbf{U}], \quad (33)$$

if

$$\partial_\mu K^\mu = 2 \left[ \mathbf{E} + \frac{m}{q} \mathbf{Q} \right] \cdot \left[ \mathbf{B} + \frac{m}{q} \mathbf{R} \right] = 0. \quad (34)$$

This is indeed the case, and can be readily verified by using the vector part of (24).

There thus exists an invariant for each dynamical species of the plasma; there is one for electrons and one for the ions for a two-component electron ion system. The ‘‘helicity’’ invariant found in Eqs. (30) and (31) is extremely general; it pertains for arbitrary temperatures and flow speeds. To the best of my knowledge, this result has not been derived before although its limiting case for nonrelativistic temperatures ( $f = 1$ ) is well known [3]. Using the conventional techniques, it would take much effort and algebra to derive a result of this generality; the new formalism makes it accessible in a few well-defined steps.

We would like to remark here that for the general isentropic fluid  $\partial_\mu K^\mu$  is not zero. Barring the special cases of either uniform temperature or density it is not even a full three divergence. Thus but for the three exceptional cases enumerated earlier,  $G$  is not a constant of the motion.

If  $m/q \rightarrow 0$  for a given species, then the invariant associated with its motion simply reverts to the standard helicity  $h$ ; this is often done in two-fluid theories where the electron inertia is neglected. In these theories, the ionic motion does, indeed, create the additional generalized helicity defined by (32) [3].

I will now derive a nonrelativistic result of considerable value and significance by exploiting the unified magnetofluid field. The equation of motion (24) can be broken into its scalar and vector parts as  $[\mathbf{U} = \gamma \mathbf{V}]$ ,

$$\mathbf{V} \cdot \hat{\mathbf{E}} = 0, \quad (35)$$

$$\hat{\mathbf{E}} + \mathbf{V} \times \hat{\mathbf{B}} = 0, \quad (36)$$

where  $\hat{\mathbf{E}} = \mathbf{E} + (m/q)\mathbf{Q}$ , and  $\hat{\mathbf{B}} = \mathbf{B} + (m/q)\mathbf{R}$  are the effective electric and magnetic fields. The fluid velocity takes the form

$$\mathbf{V} = \hat{\mathbf{e}}V + \mathbf{V}_\perp, \quad (37)$$

with

$$\mathbf{V}_\perp = \frac{\hat{\mathbf{E}} \times \hat{\mathbf{B}}}{|\hat{\mathbf{B}}|^2}, \quad (38)$$

where  $\hat{\mathbf{e}}$  is the unit vector along  $\hat{\mathbf{B}}$ . For a nonrelativistic plasma, i.e., for  $V \ll 1$ , and  $|\mathbf{V}_\perp| \ll 1$ ,  $|\hat{\mathbf{E}}| \ll |\hat{\mathbf{B}}|$ . In this approximation, the effective Lagrangian density for the magnetofluid field,

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} M^{\mu\nu} M_{\mu\nu} = -\frac{1}{2} [\hat{\mathbf{B}}^2 - \hat{\mathbf{E}}^2], \quad (39)$$

reduces to

$$\mathcal{L}_{\text{eff}}^{\text{N.R.}} \simeq -\frac{1}{2} \hat{\mathbf{B}}^2 \simeq -\frac{1}{2} \left( \mathbf{B} + \frac{m}{q} \nabla \times \mathbf{V} \right)^2, \quad (40)$$

from which it is easy to deduce that the total effective Hamiltonian (in the same approximation) is simply

$$H = \frac{1}{2} \int d^3x \left( \mathbf{B} + \frac{m}{q} \nabla \times \mathbf{V} \right)^2, \quad (41)$$

i.e., it is proportional to the effective ion-entropy or the effective energy of the magnetic field seen by the ions. The association of the ion-entropy with an effective Hamiltonian strongly fortifies the case for using it as a minimizing functional for the derivation of self-organized magnetofluid states [4].

This formalism and Eq. (24) are likely to provide new insights into the dynamics of relativistic high-temperature charged fluids. It is also hoped this compact and encompassing formalism may help guide the formulation of a similar theory for the interaction of fluids with non-Abelian gauge fields.

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