

Quantum Cosmology and Conformal Invariance

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According to Belinsky, Khalatnikov, and Lifshitz, gravity near a spacelike singularity reduces to a set of decoupled one-dimensional mechanical models at each point in space. We point out that these models fall into a class of conformal mechanical models first introduced by de Alfaro, Fubini, and Furlan (DFF). The deformation used by DFF to render the spectrum discrete corresponds to a negative cosmological constant. The wave function of the Universe is the zero-energy eigenmode of the Hamiltonian, or the spherical vector of the representation of the conformal group $SO(1,2)$. A new class of conformal quantum mechanical models with enhanced ADE symmetry is constructed, based on the quantization of nilpotent coadjoint orbits.

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While cosmological singularities have thus far eluded any satisfactory treatment in quantum gravity, insight into their generic properties is afforded by the classical analysis of Belinsky, Khalatnikov, and Lifshitz [1]. Under a self-consistent hypothesis of decoupling of the dynamics at nearby points upon approaching a spacelike singularity, they show that a generic solution of four-dimensional Einstein gravity exhibits a chaotic oscillatory behavior near the big bang (or big crunch). This subject has attracted much recent attention, with the discovery that the chaotic properties of the dynamics are tied to the hyperbolicity of the Kac-Moody algebra underlying the billiard geometry on which the motion takes place [2,3]. This is, in particular, the case of all models descending from eleven-dimensional supergravity or $N = 1$ supergravity in ten dimensions [4]. The spatial gradients, although subleading at the singularity, are also described by the Kac-Moody structure [5].

Our aim in this Letter is twofold. First, we point out an important feature of the one-dimensional model for the gravity modes at each decoupled point: It exhibits the conformal invariance found in one-dimensional quantum mechanics by de Alfaro, Fubini, and Furlan (DFF) [6].

Conformal quantum mechanical models have already appeared in black hole physics [7,8], yet their relevance to cosmology seems to have been hitherto unnoticed. More specifically, the Wheeler-DeWitt equation is but the Schrödinger equation considered by DFF, restricted to zero energy. The wave function of the universe thus lies at the bottom of a continuum of delta-function normalizable states. Retaining the effect of a negative cosmological constant discretizes the spectrum while preserving conformal invariance. In mathematical terms, the wave function of the universe is therefore the spherical vector of the representation of the conformal group. Second, we construct a new class of conformal quantum mechanical models, where the conformal group is enhanced to an ADE noncompact group. In these models, the spherical vector, and, hence, the wave function of the universe, is known exactly.

We start with $(n + 1)$ -dimensional Einstein gravity with a cosmological constant and reduce down to a $(0 + 1)$ -dimensional system. Parametrizing the metric as

$$ds^2 = -\left[\frac{\eta(t)}{V(t)}\right]^2 dt^2 + V^{2/n}(t)\hat{g}_{ij}(t)dx^i dx^j, \quad (1)$$

the Einstein-Hilbert action reduces to

$$\int dt d^n x \sqrt{-g}(R - 2\Lambda) = \int dt \left\{ \frac{1}{2\eta} \left[-\frac{2(n-1)}{n} \dot{V}^2 + V^2 \dot{U}^M G_{MN} \dot{U}^N \right] - 2\Lambda \eta \right\}, \quad (2)$$

where V denotes the volume of the spatial metric and U^M coordinatize the symmetric space $S = SI(n)/SO(n)$, with constant negative curvature, and homogeneous metric $dU^M G_{MN} dU^N := -\frac{1}{2} d\hat{g}_{ij} d\hat{g}^{ij}$ (with $\det \hat{g} = 1$). In this form, we recognize the action for a “fictitious” point particle, with mass squared $m^2 = 4\Lambda$, tachyonic for negative (AdS) Λ , propagating on a Lorentzian cone with base S and metric

$$d\sigma^2 = -\frac{2(n-1)}{n} dV^2 + V^2 dU^M G_{MN} dU^N. \quad (3)$$

The rescaled lapse η plays the role of an einbein gauge field enforcing invariance under general time reparametrizations. The appearance of the volume V with a negative kinetic term suggests its use as a “cosmological time.”

Indeed, it develops with time as $[(V/\eta)d/dt]^2 V = [2n\Lambda/(n-1)]V$, while the particle follows geodesics on the cone and, hence, on its base S . A similar reduction in the presence of extra scalar and gauge fields would yield a cone over an enlarged homogeneous space [e.g., $SO(n, n)/[SO(n) \times SO(n)]$ in the presence of a Kalb-Ramond two form and dilaton].

Since the conical moduli space (3) admits a homothetic Killing vector $V\partial_V$, the free particle should exhibit conformal invariance [8]. This is easily shown by introducing conjugate momenta p and P_M for the canonically normalized volume coordinate $\rho = \sqrt{8(n-1)V/n}$ and the shape moduli U^M . The Hamiltonian following from the action (2) reads

$$H = \frac{\eta}{V} \left[\frac{1}{2} p^2 + \frac{4(n-1)}{n\rho^2} \Delta - \frac{n\Lambda}{4(n-1)} \rho^2 \right]. \quad (4)$$

The equation of motion of η forces H to vanish, after which the gauge $\eta = V$ can be imposed. Here $\Delta = -P_M G^{MN} P_N$ is the quadratic Casimir of the action of $Sl(n)$ on the homogeneous space $Sl(n)/SO(n)$ and has vanishing Poisson brackets with ρ and p . Therefore it effectively plays the role of a coupling constant $g = 8(n-1)\Delta/n$ for the $1/\rho^2$ potential. Indeed, at $\Lambda = 0$, we recognize in (4) the Hamiltonian of the conformal mechanical system introduced by de Alfaro, Fubini, and Furlan [6]. The generators,

$$E_+ = \frac{1}{2} \rho^2, \quad D_0 = \frac{1}{2} \rho p, \quad E_- = \frac{1}{2} \left(p^2 + \frac{g}{\rho^2} \right), \quad (5)$$

represent the conformal group $SO(1, 2)$ in $(0+1)$ dimensions, $\{E_+, E_-\} = 2D_0$, $\{D_0, E_\pm\} = \pm E_\pm$. For vanishing Λ , the Hamiltonian $H = E_-$ has unit dimension with respect to the generator of conformal rescalings D_0 ; hence, the system is conformally invariant. Remarkably, the introduction of a cosmological term preserves the action of the conformal group $SO(2, 1)$, as it simply amounts to choosing a different generator $H = E_- - n\Lambda/(2n-2)E_+$ as the Hamiltonian. This deformation was considered in [6] although without a clear physical motivation.

Let us now discuss some aspects of the quantum dynamics in this toy model of a cosmological singularity. While we understand little about quantum gravity at a spacelike singularity, we may assume that the decoupling of nearby points still holds and works in a minisuperspace truncation. Replacing canonical momenta by their Schrödinger representation $p \rightarrow i\partial/\partial\rho$, $P_M \rightarrow i\partial/\partial U^M$, the Hamiltonian constraint (4) becomes the Wheeler-DeWitt equation [9],

$$H\psi = \left(-\frac{1}{2} \partial^2 + \frac{4(n-1)}{n\rho^2} \Delta - \frac{n\Lambda}{4(n-1)} \rho^2 \right) \psi = 0, \quad (6)$$

acting on wave functions $\psi(\rho, U^M)$, where Δ is the quadratic Casimir of the $Sl(n)$ action on the homogeneous space S . As usual, a vanishing Hamiltonian implies that the wave function ψ is independent of the time t ; never-

theless correlations between the volume ρ and the other observables U_M may be used to set up measurements [9]. One may now recognize the Wheeler-DeWitt Eq. (6) as the Schrödinger equation of DFF's conformal quantum mechanics, with coupling $g = 8(n-1)\Delta/n$, restricted to zero-energy states. Invariance under the conformal group $SO(1, 2)$ is retained at the quantum level after resolving the ordering ambiguity $D_0 \rightarrow (\rho p + p\rho)/4$. (Indeed, the requirement of conformal symmetry and its extension below uniquely fix all ordering ambiguities.) In particular, the effect of a cosmological constant $\Lambda < 0$ is to replace the parabolic generator E_- with continuous spectrum \mathbb{R}^+ , by a compact, discrete spectrum, generator $H = E_- - n\Lambda/(2n-2)E_+$ (for $\Lambda > 0$, H has a continuous spectrum). It is intriguing that this seemingly favorable case corresponds to a *tachyonic* fictitious particle.

Despite the formal identity between the WDW and DFF Hamiltonians, it is worth noting several crucial differences. First, the WDW equation picks out zero-energy modes of H only, so that the requirement of boundedness from below is no longer necessary. Indeed, the coupling g in our problem appears to be negative on square integrable wave functions on S (for which $\Delta < 0$), and so does the mass term for $\Lambda < 0$. This is in fact a standard feature of canonical gravity, where the conformal factor always appears with a kinetic term of the “wrong” sign [9]. Similarly, in a traditional quantum mechanics setup, one usually requires states to have a finite L_2 norm around $\rho = 0$ [6]. When ρ is viewed as a cosmological time, the requirement of square normalizability is no longer sensible (it can, however, be useful to select recollapsing universes [9]). The analogy of (6) with a massive Klein-Gordon equation would suggest instead to consider the Klein-Gordon norm on spacelike slices of fixed ρ (“third” quantization may be used to cure the nonpositive definiteness [10], although its interpretation remains unclear). At any rate, our current understanding of cosmological singularities does not allow us to specify the boundary conditions reliably; we therefore proceed without further ado.

A few comments are in order about the chaotic properties of these cosmological models. First, our discussion was carried out for free Kasner flights, in the absence of the potential terms coming from spatial gradients. As one approaches the spacelike singularity, the potential terms behave as infinitely steep reflection walls [11]. It is possible that their effect could be mimicked by modding out by a discrete symmetry group [e.g., the walls exchanging the various radii are included in a $Sl(n, \mathbb{Z})$ subgroup of $Sl(n)$ acting on S]. This group is, however, of a rather wild nature, as it should contain the Weyl group of a hyperbolic Kac-Moody algebra [2,3], and it is unclear at this stage how to describe the geodesic motion on such an object. One may speculate that the universal $SO(2, 1)$ subalgebra uncovered in [12] in general hyperbolic Kac-Moody algebra may play a role in implementing conformal invariance.

We now present a construction of a class of conformally invariant quantum systems based on the quantization of nilpotent coadjoint orbits of finite Lie groups. In contrast to generic orbits, nilpotent ones are especially interesting as they possess fewer or no free parameters. Such conformal systems were first found in the course of constructing theta series for nonsymplectic groups [13], and a particular example was given independently in [14] for the minimal representation of E_8 . For simplicity, we shall illustrate it on the simplest nontrivial case, D_4 , which will coincide with the model (6) for $(2 + 1)$ gravity.

The classical phase space of our systems arises from the coadjoint orbit of a nilpotent element of smallest order in a finite simple Lie algebra G . This element can be conjugated into the generator associated to the lowest root $E_{-\omega}$. The $Sl(2)$ subalgebra generated by $\{E_\omega, D_\omega, E_{-\omega}\}$ with $D_\omega = [E_\omega, E_{-\omega}]$ will be the conformal group in $(0 + 1)$ dimensions, and $E_{-\omega}$ be chosen as the Hamiltonian. The Cartan generator D_ω grades the algebra into five subspaces, $G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus G_2$, such that the top and bottom subspaces are one dimensional, $G_{\pm 2} = \mathbb{R}E_{\pm\omega}$. The coadjoint orbit of $E_{-\omega}$ can be parametrized by $P \backslash G = \mathbb{R}H_\omega \oplus G_1 \oplus G_2$, where P is the stabilizer of $E_{-\omega}$ under the coadjoint action. The level-one space G_1 is a Heisenberg algebra, which can be diagonalized in the form $[E_{\beta_i}, E_{\gamma_j}] = \delta_{ij}E_\omega$. We can thus represent these generators as canonical coordinates and conjugate momenta, $E_{\beta_i} = yp_i$, $E_{\gamma_i} = x_i$, $E_\omega = y$. The maximal subalgebra H in G commuting with $SO(2, 1)$ lies in G_0 and acts linearly as canonical transformations on the coordinates and momenta $\{x_i, p_i\}$, leaving y invariant. The choice of a polarization of G_1 into coordinates and momenta further breaks H to a subgroup H_l acting linearly on the coordinates. A standard polarization is to choose for β_0 the simple root to which the affine root ω attaches on the Dynkin diagram of G and for $\beta_{i>0}$ the positive roots such that $\langle \beta_0, \beta_i \rangle = 1$ [15]. The canonical momentum associated to y is obtained in turn from the Cartan generator associated to β_0 , $D_{\beta_0} = [E_{\beta_0}, E_{-\beta_0}] = py - p_0x_0$. So as to bring the generator E_ω to the conformal quantum mechanics form (5), we make a canonical transformation,

$$y = \frac{\rho^2}{2}, \quad x_i = \frac{\rho q_i}{2}, \quad p = \frac{1}{\rho}p_\rho - \frac{1}{\rho^2}q_i\pi_i, \quad p_i = 2\frac{\pi_i}{\rho}. \quad (7)$$

The generators of the $SO(2, 1)$ subalgebra take the form

$$E_\omega = \frac{1}{2}\rho^2, \quad D_\omega = \rho p_\rho, \quad E_{-\omega} = \frac{1}{2}\left(p^2 + \frac{4\Delta}{\rho^2}\right), \quad (8)$$

where Δ is, up to an additive constant, the quadratic Casimir of H , hence, a quartic invariant of the coordinates and momenta $\{q_i, \pi_i\}$. Finally, the symmetry is enhanced from $SO(2, 1) \times H$ to all of G using two discrete generators: (i) the Fourier transform on all positions q_i

at once, corresponding to the longest word in the Weyl group, and (ii) the Weyl reflection with respect to β_0 , acting on wave functions as $W\psi(\rho, q_0, q_i) = e^{-[I_3(q_i)]/(2q_0)}\psi(\sqrt{-\rho q_0}, \sqrt{-\rho^3/q_0}, \sqrt{-\rho q_i/q_0})$, where I_3 is the cubic invariant of the positions under the linearly realized H_l . The compatibility between the two actions, and indeed the whole construction, relies heavily on the invariance of the non-Gaussian character $\exp[iI_3(x_i)/x_0]$ under Fourier transform [15].

As an example, the algebra D_4 decomposes as $1_{-2} \oplus (2, 2, 2)_{-1} \oplus [(1, 1, 1) + (3, 1, 1) + \text{perm}]_0 \oplus (2, 2, 2)_1 \oplus 1_2$ under $Sl(2)^3 \times \mathbb{R}$, where \mathbb{R} denotes the Cartan generator D_ω associated to the highest root. The coordinates and momenta correspond to the grade-one space and transform as a $(2, 2, 2)$ of $H = Sl(2)^3$. They satisfy the Heisenberg algebra $[q^{aA\alpha}, q^{bB\beta}] = \epsilon^{ab}\epsilon^{AB}\epsilon^{\alpha\beta}$. The actions of each $Sl(2)$ factor in H are represented by the angular momentumlike operators, $\Sigma^\mu = \sigma_{\alpha\beta}^\mu \epsilon_{ab}\epsilon_{AB}q^{aA\alpha}q^{bB\beta}$, with similar definitions for s^m and S^M . It is easy to check that these generators satisfy the $Sl(2)$ algebra, $[\Sigma^\mu, \Sigma^\nu] = \epsilon^{\mu\nu\rho}\Sigma^\rho$. The quadratic Casimirs of all three $Sl(2)$ are identical and equal to the unique quartic invariant of the $(2, 2, 2)$ representation. The generators of the $SO(2, 1)$ subalgebra then read as in (8) with Δ the common quadratic Casimir of the three $Sl(2)$. To clarify the meaning of the Hamiltonian in (8), let us choose a polarization such that the $Q^{A\alpha} = q^{1A\alpha}$ are coordinates and $q^{2A\alpha}$ are momenta. The bispinor $Q^{A\alpha}$ can be thought of as a vector Q^I of $SO(2, 2)$, parametrized by three ‘‘polar angles’’ $\Omega \in H_3 = SO(2, 2)/SO(2, 1) = SO(2, 1)$ and its length squared $\kappa^2 = Q^I\eta_{IJ}Q^J$, where η_{IJ} is the signature $(2, 2)$ metric. The quadratic Casimir then corresponds to the angular momentum squared on the pseudosphere H_3 , i.e., the Laplacian on $Sl(2)$. Out of the four coordinates Q^I , we see that only the three hyperbolic polar angles receive kinetic terms in the Hamiltonian $E_{-\omega}$, while the radius κ decouples.

A natural object in the theory of minimal representations is the spherical vector, i.e., the wave function annihilated by all compact generators $E_\alpha + E_{-\alpha}$. For irreducible and so-called spherical representations, this vector is unique. From the expression (8), we see that a spherical vector corresponds to a zero-energy state of the Hamiltonian with a negative ‘‘cosmological constant,’’ i.e., $H = E_{-\omega} + E_\omega$. For D_4 , the spherical vector should also be invariant under the compact $U(1)$ inside each $Sl(2)$ factor, so the coordinates of the D_4 conformal quantum mechanics are effectively valued in $H_3/U(1) = Sl(2)/U(1)$. This is indeed the conformal model (6) of $(2 + 1)$ gravity with $S = SO(2)$. Therefore, supplementing the fields $(\rho; U_1, U_2) \in \mathbb{R} \times Sl(2)/U(1)$ with an angular variable and a decoupled radius κ , the conformal symmetry of reduced $D = 2 + 1$ gravity is enhanced to an $SO(4, 4)$ noncompact spectrum generating symmetry. The spherical vector of the D_4 minimal representation has been obtained in [13]; in the polarization used here it reads

TABLE I. Nonlinearly realized symmetry G , canonically realized H , representation of positions and momenta under H , linearly realized H_l , homogeneous space S including decoupled factors \mathbb{R}_κ , and functional dimension of the Hilbert space for models with enhanced conformal symmetry.

G	H	G_1	H_l	S	Dim
A_{n-1}	$\text{Gl}(n-2)$	$[n-2] + [n-2]$	$\text{Sl}(n-2)$	$\mathbb{R}_\kappa \times \text{Sl}(n-2)/[\text{Sl}(n-3) \times \mathbb{R}^{n-3}]$	$n-1$
D_n	$\text{Sl}(2) \times D_{n-2}$	$(2, 2n-4)$	D_{n-2}	$\mathbb{R}_\kappa \times \text{SO}(n-2, n-2)/\text{SO}(n-3, n-2)$	$2n-3$
E_6	$\text{Sl}(6)$	$20 = [0, 0, 1, 0, 0]$	$\text{SO}(3, 3)$	$\mathbb{R}_\kappa \times \text{SO}(3, 3)/[\text{SO}(3) \times \text{SO}(3)]$	11
E_7	$\text{SO}(6, 6)$	32	$\text{Sl}(6)$	$\mathbb{R}_\kappa \times \text{Gl}(6)/\text{Sp}(6)$	17
E_8	E_7	56	$\text{Sl}(8)$	$\mathbb{R}_\kappa \times \text{Sl}(8)/\text{Sp}(8)$	29

$$\psi_{D_4} = \frac{\rho^{3/2} e^{-S}}{S}, \quad S = \frac{1}{2} \sqrt{\rho^4 + \rho^2 \text{tr}(Q'Q) + \kappa^4}. \quad (11)$$

It would be very interesting if the new coordinate $\kappa^2 = \det(Q^{A\alpha})$, appearing here as a degeneracy label, had a cosmological interpretation.

A similar conformal quantum mechanical system can be constructed for any finite simple Lie algebra G , except for the $\text{SO}(2n+1)$ series. The details of the construction of the minimal representation have been spelled out for the simply laced case in [13], where the spherical vector has also been obtained. In order to translate the results of [13] into the presentation (8) suitable for interpreting $E_{-\omega}$ as the Hamiltonian of a quantum mechanical system, one only needs to perform the canonical transformation (7). We thus obtain a class of quantum mechanical systems with an enhanced conformal symmetry G corresponding to any finite simple Lie group in (noncompact) split real form. The field content and symmetries of these models are summarized in Table I. A similar construction can also be carried out for other (non-nilpotent) orbits, or for the nonsimply laced groups G_2 and F_4 , whose minimal representation does not possess any spherical vector, so that part of the compact symmetries are spontaneously broken. An open problem is to identify gravitational theories reducing to those in the process of dimensional reduction.

In conclusion, conformal quantum mechanics is relevant to the dynamics of gravity at a spacelike singularity. A negative cosmological constant renders the spectrum of the Wheeler-DeWitt operator discrete. The wave function of the universe is then obtained as the spherical vector of the representation of the enhanced conformal group. We have constructed a family of conformal quantum mechanical models with symmetry enhanced to an arbitrary simple noncompact group G in split real form. For $G = D_4$, this reproduces the conformal mechanics arising from the reduction of $(2+1)$ -dimensional gravity, together with a decoupled field κ yet to be understood. An important question is whether this construction can be extended to infinite Kac-Moody groups such as E_{10} or

BE_{10} , which should control the dynamics of M theory or the heterotic string at a spacelike singularity. It would be exciting if the dynamics of gravity at a spacelike singularity was related to the geodesic motion on a coadjoint orbit of E_{10} , in analogy with conventional incompressible fluid hydrodynamics.

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